THE SIMPLE HARMONIC OSCILLATOR

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ABSTRACT. An exploration of the non-square-integrable eigenfunctions of the quantum simple harmonic oscillator. Unlike the square-integrable eigenfunctions, these form a continuous spectrum, and in fact a Riemann surface. The eigenfunctions themselves are given by the confluent hypergeometric series (Kummer's function).

1. INTRODUCTION

Like what the abstract says

2. SOLUTIONS

Introduction to basic notation. The classical hamiltonian:

$$\mathcal{H}=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2$$

Quantized by taking

$$p=-i\hbar\frac{d}{dx}$$

with classic square-integrable eigenfunctions forming a Hilbert space:

$$\psi_n(x)=\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

where n is an integer, and H_n are the Hermite polynomials

$$H_n(y)=(-1)^n\exp\left(y^2\right)\frac{d^n}{dy^n}\exp\left(-y^2\right)$$

There are eigenfunctions:

$$\mathcal{H}\psi_n=\left(n+\frac{1}{2}\right)\psi_n$$

It is convenient to eliminate the extraneous constants. Writing

$$y=\sqrt{\frac{m\omega}{\hbar}}x$$

one then has that the Hamiltonian is given by

$$H=\frac{\mathcal{H}}{\hbar\omega}=\frac{1}{2}\left(-\frac{d^2}{dy^2}+y^2\right) \tag{2.1}$$

The remainder of this paper will be devoted to exploring the continuous-spectrum solutions of this normalized Hamiltonian, that is, discovering and describing the solutions $\psi(y)$ of

$$H\psi=\lambda\psi \tag{2.2}$$

for $\lambda\in\mathbb{C}$ the complex plane.

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3. Confluent Hypergeometric Solutions

This section derives the confluent hypergeometric eigenfunctions. Let

$$\psi(y) = \exp\left(-\frac{y^2}{2}\right)\varphi(y)$$

Then the differential equation becomes

$$\varphi'' - 2y\varphi' + (2\lambda - 1)\,\varphi = 0$$

Substituting $z = y^2$ and $\eta(z) = \varphi(y)$, one obtains the differential equation

$$z\eta'' + \left(\frac{1}{2} - z\right)\eta' - \left(\frac{1-2\lambda}{4}\right)\eta = 0$$

which may be immediately recognized as Kummer's differential equation for the confluent hypergeometric functions:

$$z\eta'' + (b-z)\,\eta' - a\eta = 0$$

which has solutions

$$M(a,b;z) = {}_1F_1(a,b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

and

$$U(a,b;z) = \frac{\pi}{\sin\pi b}\left[\frac{M(a,b;z)}{\Gamma(1+a-b)\Gamma(b)} + z^{1-b}\frac{M(1+a-b,2-b;z)}{\Gamma(a)\Gamma(2-b)}\right]$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the rising factorial. From the point of view of presenting solutions to equation 2.2, there are only two linearly independent solutions. These are

$$\psi_1(y) = e^{-y^2/2}M\left(\frac{1-2\lambda}{4}, \frac{1}{2}; y^2\right)$$

and

$$\psi_2(y) = y\,e^{-y^2/2}M\left(\frac{3-2\lambda}{4}, \frac{3}{2}; y^2\right)$$

The appropriate normalization for these two solutions is not yet clear, since neither is square-integral along the real y-line, except when $\lambda = n+1/2$, of course.

A few symmetries may be noted. The Hamiltonian 2.1 changes sign under the substitution $y \to iy$, and so the eigenvalues flip sign: $\lambda \to -\lambda$. Thus, "rotating" the eigenfunctions by 90 degrees corresponds to flipping the sign of the eigenvalue.

4. Ladder Operators

The traditional ladder operators for the simple harmonic oscillator may be written as

$$a^\dagger = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right)$$

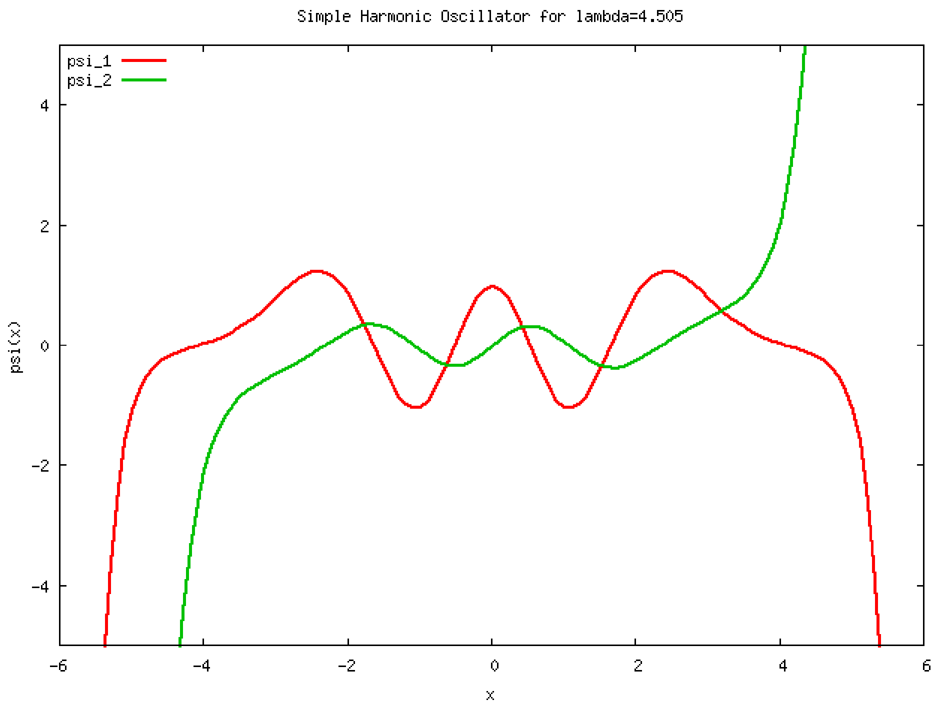
for the raising operator, and

$$a = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right)$$

for the lowering operator, so that

$$H = \frac{1}{2}\left(-\frac{d^2}{dy^2} + y^2\right) = a^\dagger a + \frac{1}{2}$$

FIGURE 3.1. Graph of ψ_1 and ψ_2



This line graph shows $\psi_1(x)$ and $\psi_2(x)$ for x real, and $\lambda = 4.505$. This eigenvalue is very near 4.5, at which point ψ_1 would be a product of the Hermite polynomial $H_4(x)$ and $\exp -x^2/2$. However, being slightly off ths square-integrable eigenvalue, the resulting eigenfunctions diverge exponentially outside of this narrow oscillatory region.

Under the action of these operators, one has that

$$a\psi_1(\lambda) = \sqrt{2}\left(\frac{1}{2} - \lambda\right)\psi_2(\lambda - 1)$$

and

$$a\psi_2(\lambda) = \frac{1}{\sqrt{2}}\psi_1(\lambda - 1)$$

Note that the lowering operator has only one function in its kernel, namely ψ_1 for $\lambda = 1/2$; this is the ground state of the traditional harmonic oscillator. For all other eigenvalues, the lowering operator is a bi-directional shift operator. A similar observation applies for the rasing operator, for which one has

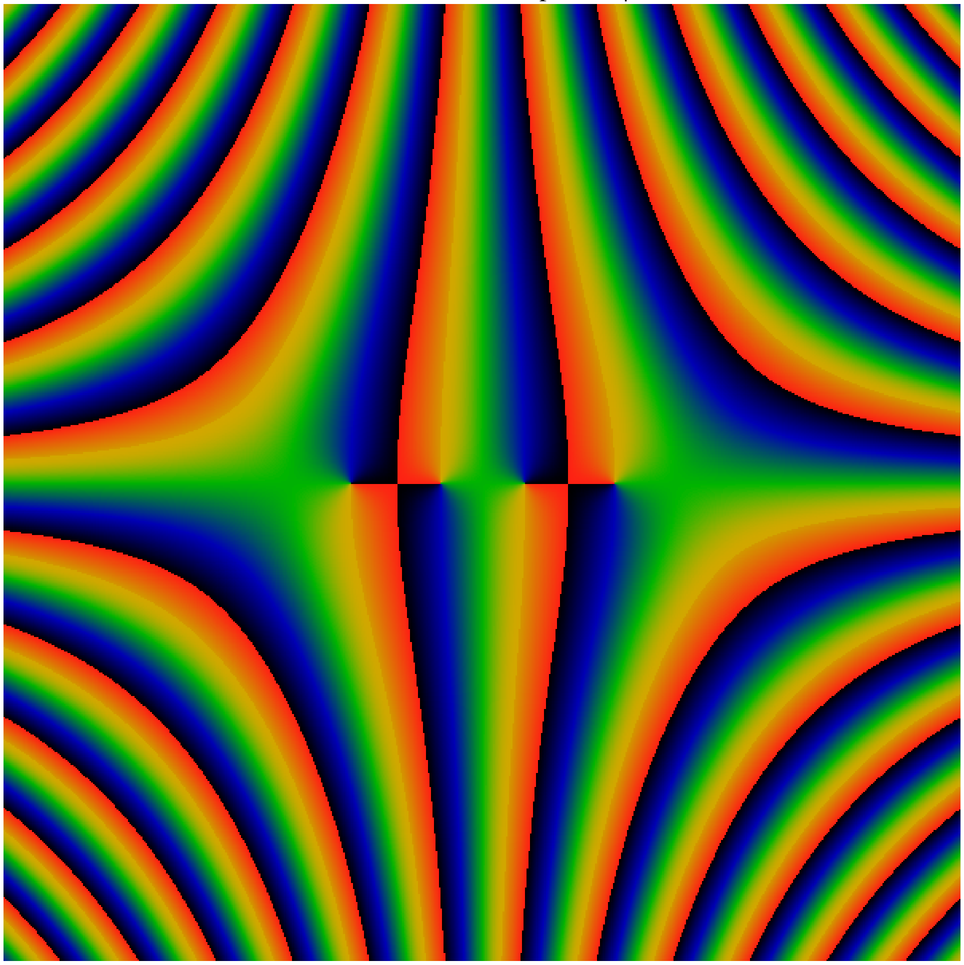
$$a^\dagger\psi_1(\lambda) = \sqrt{2}\left(\lambda + \frac{1}{2}\right)\psi_2(\lambda + 1)$$

and

$$a^\dagger\psi_2(\lambda) = -\frac{1}{\sqrt{2}}\psi_1(\lambda + 1)$$

and so again, the operator is a bi-directional shift, with only one function in its kernel, namely ψ_1 for $\lambda = -1/2$.

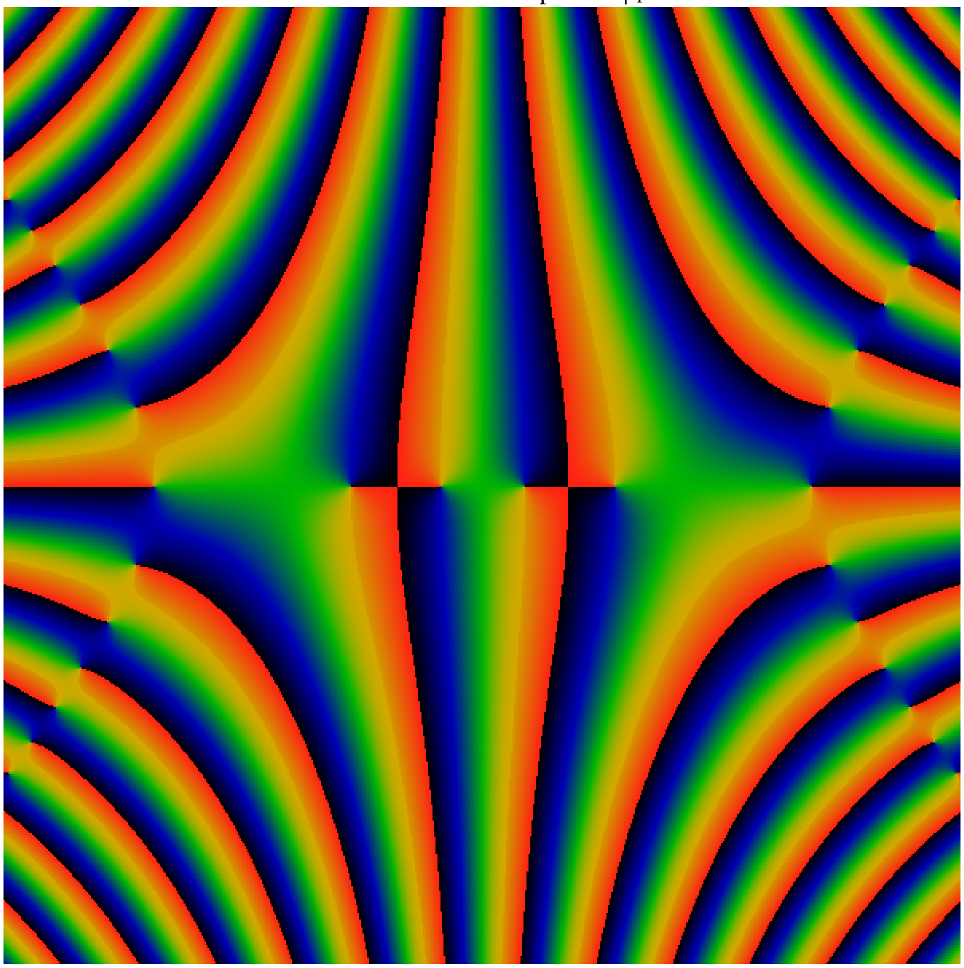
FIGURE 3.2. Phase plot of ψ_1



This figure shows a phase plot of $\psi_1(y)$ for $\lambda = 4.5$, which corresponds to the integer eigenvalue of $n = 4$ for the standard quantum harmonic oscillator. The square represents values of y on the complex plane, bounded by the square $-6 \le \Re y \le 6$ and $-6 \le \Im y \le 6$. The colors denote values of $\arg\psi_1$, with black representing $\arg\psi_1 = -\pi$, green $\arg\psi_1 = 0$ and red a phase of $+\pi$. Zeros are clearly visible as points around which the full spectrum of colors wrap around. This images shows four zeros, arranged in the real axis, which are located exactly at the zeros of the Hermite polynomial $H_4(y)$. As the image suggests, there are no other zeros on the complex y plane. The images for other values of $\lambda = n + 1/2$ for positive integer values of n are qualitatively similar, with the exception that addtional zeros appear on the real axis, along with similarly appropriate asymptotic behaviour along the imaginary axis,

The figure for $\lambda = -4.5$ is identical to this, except that it is rotated by 90 degrees. This is because a subssistution of $y \to iy$ takes $\lambda \to -\lambda$.

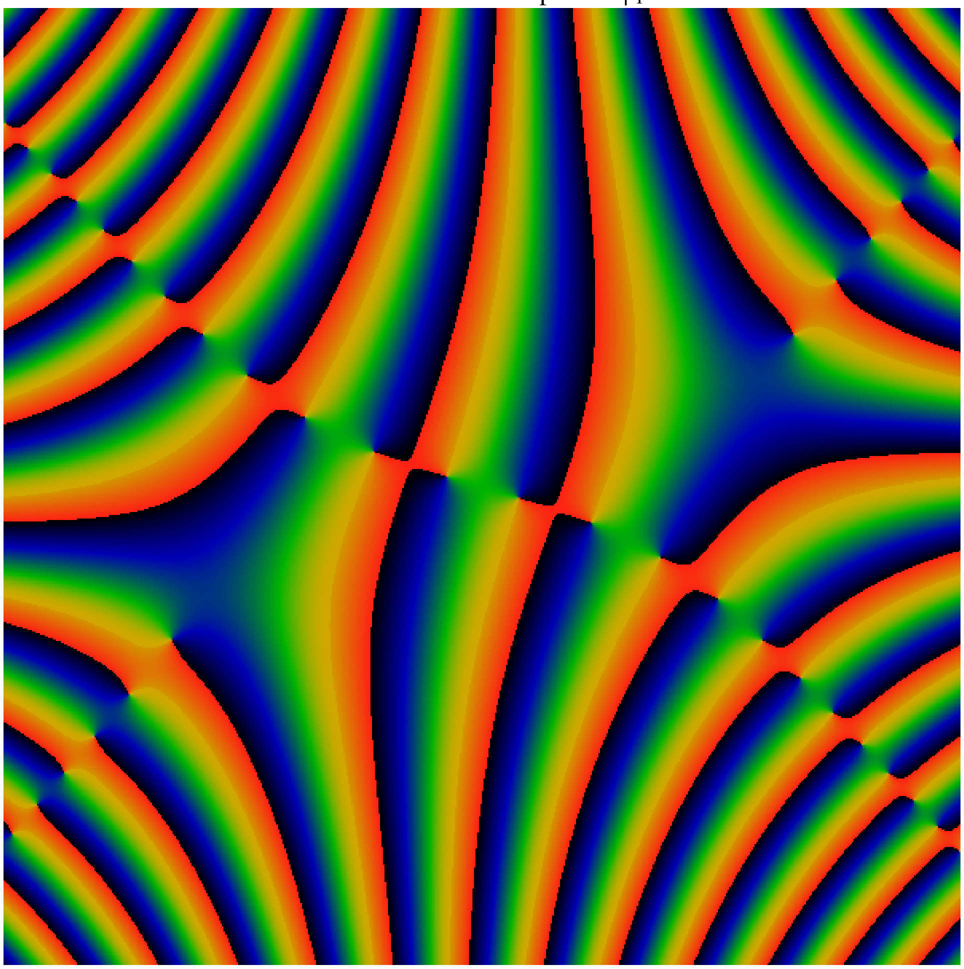
FIGURE 3.3. Phase plot of ψ_1



This figure shows a phase plot of $\psi_1(y)$ for $\lambda = 4.505$. The domain and color scheme are exactly as those for the image 3.2. Unlike image for $\lambda = 4.5$, there are additional zeros. Two appear on the real axis, and, more notably, a hyperbola-like arrangement of zeros asymptotically approaching the 45-degree diagonals. As the graphic suggests, there are an infinite number of these. The asymptotes can be explained by noting that $\exp(y^2/2)$ behaves like the sine function along the $y = r\exp \pm i\pi/4$ diagonals. The images for other real values of λ are qualitatively similar, provided that λ is not a half-integer. It should be noted that the appearance of the zeros at the diagonals is completely discontinuous as a function of λ, in that the figure for $\lambda = 4.5 + \varepsilon$ for arbitrarily small $\varepsilon \neq 0$ will look essentially like this figure, while that for $\varepsilon = 0$ will be that of figure 3.2.

Although these eigenfunctions cannot be square-normalized to give them a "natural" scale, they can be scaled so that they appear more symmetric under the action of the ladder

FIGURE 3.4. Phase plot of ψ_1



This figure shows a phase plot of $\psi_1(y)$ for $\lambda = 4.6 + i3.0$. The domain and color scheme are exactly as those for the image 3.2. Unlike image for $\lambda = 4.505$, there are no longer any zeros on the real axis. In a qualtitative sense, the number of zeros have remained the same; the location have changed. The 45-degree diagonal asymptotes appear to be unaltered. Figures for similar nearby values of λ are qualitatively similar.

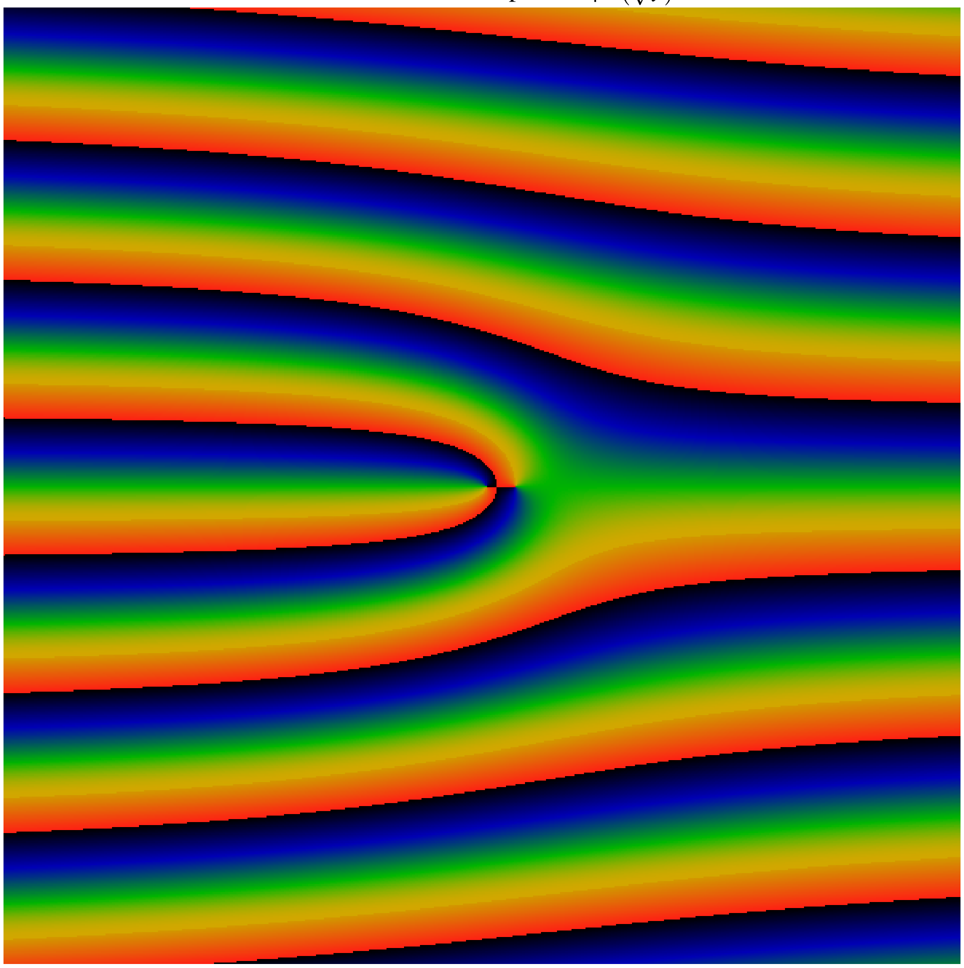
operators. Write

$$\chi_1(\lambda) = \Gamma\left(\frac{2\lambda+1}{4}\right)\sqrt{\frac{2^\lambda}{\Gamma\left(\frac{2\lambda+1}{2}\right)}}\exp\left(\frac{i\pi\lambda}{2}\right)\psi_1(\lambda)$$

and

$$\chi_2(\lambda) = \Gamma\left(\frac{2\lambda-1}{4}\right)\sqrt{\frac{2^\lambda\left(\lambda-\frac{1}{2}\right)}{\Gamma\left(\frac{2\lambda-1}{2}\right)}}\exp\left(\frac{i\pi(\lambda-1)}{2}\right)\psi_1(\lambda)$$

FIGURE 3.5. Phase plot of $\psi_1\left(\sqrt{y}\right)$



This figure shows a phase plot of $\psi_1(\sqrt{y})$ for $\lambda = 4.5$. The color scheme is exactly as those for the image 3.2, while the domain is $-40 \leq \Re y, \Im y \leq 40$. The square-root was choosen out of recognition that the dominant hyperbola-like features of the previous graphs are nothing other than the typical feature of a phase plot of the square of domain. Removing this distraction more clearly shows the primary qualities of ψ_1. As in the figure 3.2, the only zeros are on the real axis. This is in sharp contrast to any value of λ which is not exactly a half-integer.

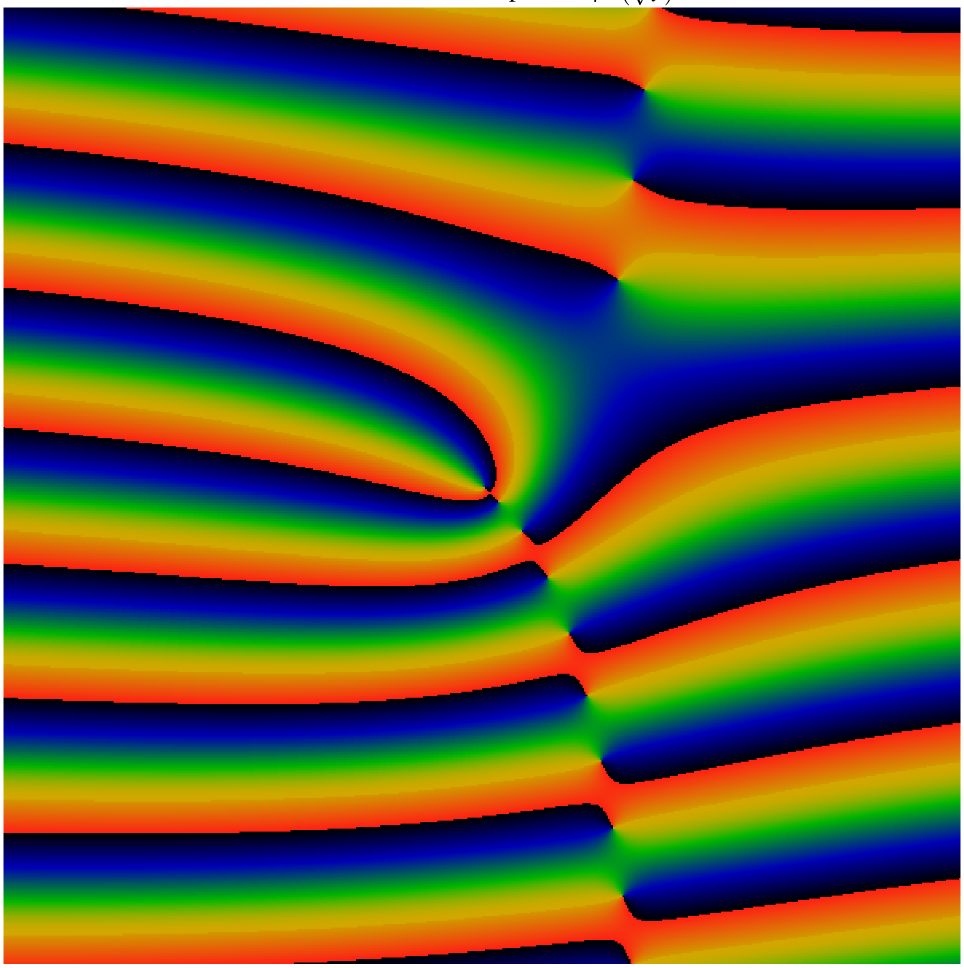
Then one has the far more symmetric relations

$$a\chi_1(\lambda) = \sqrt{\lambda - \frac{1}{2}}\chi_2(\lambda - 1)$$

and

$$a\chi_2(\lambda) = \sqrt{\lambda - \frac{1}{2}}\chi_1(\lambda - 1)$$

FIGURE 3.6. Phase plot of $\psi_1\left(\sqrt{y}\right)$



This figure shows a phase plot of $\psi_1(\sqrt{y})$ for $\lambda = 4.6 + i4.0$. The color scheme and domain is exactly as for the image 3.5. As for any λ that is not precisely a half integer, there is an arrangement of zeros along the vertical. As for any λ with an imaginary part, there are no zeros on the real axis.

Substituting $\lambda = n + 1/2$, it can be seen that these have precsely the normalization used for the canonical half-integer-valued eigenfunctions. The raising operators also have the standard form:

$$a^{\dagger}\chi_1(\lambda) = \sqrt{\lambda + \frac{1}{2}}\chi_2(\lambda + 1)$$

and similarly for the exchange $1 \leftrightarrow 2$.

5. Coherenet States

The eigenfunctions of the shift operators are known as coherent states. These are more easily constructed by using an alternate normalization for the states. Let

$$\eta_{1,2}(\lambda) = \sqrt{\Gamma\left(\lambda + \frac{1}{2}\right)}\chi_{1,2}(\lambda)$$

Then the lowering operator has the simpler form $a\eta_1(\lambda) = \eta_2(\lambda - 1)$ and $a\eta_2(\lambda) = \eta_1(\lambda - 1)$ on these states. For $\lambda \neq 1/2$, eigenstates of the shift operator may be written as

$$\phi_\theta(q) = \sum_{n=-\infty}^{\infty} q^n \left(\eta_1(\lambda+n) + e^{i\theta}\eta_2(\lambda+n)\right)$$

so that one has

$$a\phi_\theta(q) = qe^{-i\theta}\phi_\theta(q)$$

Formulated in this way, one has an explicit 2πdegeneracy in the eigenvalue spectrum. Or do we ??

6. ToDO

Monodromy
Theta function representation

7. Strange ideas

What is

$$f_{n,p}(y) = \exp(y^p)\frac{d^n}{dy^n}\exp(-y^p)$$

for general values of p???