NOTES RELATING TO NEWTON SERIES FOR THE RIEMANN ZETA FUNCTION

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ABSTRACT. This paper consists of the extended working notes and observations made during the development of a joint paper[**?**] with Philippe Flajolet on the Riemann zeta function. Most of the core ideas of that paper, of which a majority are due to Flajolet, are reproduced here; however, the choice of wording used here, and all errors and omissions are my own fault. This set of notes contains considerably more content, although is looser and sloppier, and is an exploration of tangents, dead-ends, and ideas shooting off in uncertain directions.

The finite differences or Newton series of certain expressions involving the Riemann zeta function are explored. These series may be given an asymptotic expansion by converting them to Norlund-Rice integrals and applying saddle-point integration techniques. Numerical evaluation is used to confirm the appropriateness of the asymptotic expansion. The results extend on previous results for such series, and a general form for Dirichlet L-functions is given. Curiously, because successive terms in the asymptotic expansion are exponentially small, these series lead to simple near-identities. A resemblance to similar near-identities arising in complex multiplication is noted.

1. INTRODUCTION

The binomial coefficient is implicated in various fractal phenomena, and the Berry conjecture [ref] suggests that the zeros of the Riemann zeta function correspond to the chaotic spectrum of an unknown quantization of some simple chaotic mechanical system. Thus, on vague and general principles, one might be lead to explore Newton series involving the Riemann zeta function. Of some curiosity is the series

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{b_n}{n!} (s)_n \tag{1.1}$$

where $(s)_n = s(s-1)\cdots(s-n+1)$ is the falling Pochhammer symbol. Note the general resemblance to a Taylor's series, with the Pochhammer symbol taking the place of the monomial. Such general types of resemblances are the key idea underlying the so-called "umbral calculus".

Here, the constants b_n play a role analogous to the Stieltjes constants of the Taylor expansion. The b_n have several remarkable properties. Unlike the Stieltjes constants, they may be written as a finite sum over well-known constants:

$$b_n = n(1-\gamma-H_{n-1}) - \frac{1}{2} + \sum_{k=2}^{n} (-)^k \left(\begin{array}{c} n \\ k \end{array} \right) \zeta(k) \tag{1.2}$$

for $n > 0$. Here

$$H_n = \sum_{m=1}^{n} \frac{1}{m} \tag{1.3}$$

Date: November 14, 2006.

are the harmonic numbers. The first few values are

$$b_0 = \frac{1}{2} \tag{1.4}$$

$$b_1 = -\gamma + \frac{1}{2} \tag{1.5}$$

$$b_2 = -2\gamma - \frac{1}{2} + \zeta(2) \tag{1.6}$$

$$b_3 = -3\gamma - 2 - \zeta(3) + 3\zeta(2) \tag{1.7}$$

$$b_4 = -4\gamma - \frac{23}{6} + \zeta(4) - 4\zeta(3) + 6\zeta(2) \tag{1.8}$$

Although intermediate terms in the sum become exponentially large, the values themselves become exponentially small:

$$b_n = \mathcal{O}\left(n^{1/4} e^{-2\sqrt{\pi n}}\right)$$

The development of an explicit expression for the asymptotic behavior of b_n for $n \to \infty$ is the main focus of this essay. The principal result is that

$$b_n = \left(\frac{2n}{\pi}\right)^{1/4} \exp\left(-\sqrt{4\pi n}\right) \cos\left(\sqrt{4\pi n} - \frac{5\pi}{8}\right) + \mathcal{O}\left(n^{-1/4} e^{-2\sqrt{\pi n}}\right) \tag{1.9}$$

The method of attack used to obtain this result is to convert the Newton series into an integral of the Norlund-Rice form[**?**], which may be evaluated using steepest-descent / stationary-phase methods [need ref]. These techniques are sufficiently general that they can be applied to other similar sums that appear in the literature [multiple refs] (See penultimate section below, where these are reviewed in detail). One of the principal results of this essay is to apply this asymptotic expansion to the Newton series for the Dirichlet L-functions. Additional results concern the generalization of b_n to $b(s)$ for generalized complex s, so that $b(n) = b_n$ at the integers n.

Another set of sections explore the relationship to the Stieltjes constants, and attempt to find an asymptotic expansion for these.

The development of the sections below are in no particular order; they follow threads of thoughts and ideas in various directions.

2. Derivation

This section demonstrates the relationship between the Riemann zeta function and the b_n, starting from first principles. First, the values of the b_n are derived, and then it is shown that the resulting series is equivalent to the Riemann zeta function.

The b_n may be derived by a straightforward inversion of a binomial transform. Given any sequence $\{s_n\}$ whatsoever, the binomial transform [ref Knuth concrete math] of this sequence is another sequence $\{t_n\}$, given by

$$t_n = \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} s \\ k \end{array}\right) s_k$$

The binomial transformation is an involution, in that the original series is regained by the same transform applied a second time:

$$s_n = \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} s \\ k \end{array}\right) t_k$$

A demonstration of the above is straight-forward, in part because each sum is finite.

By assuming the sequence for the Riemann zeta at the integers:

$$\zeta(n) = \frac{1}{n-1} + \sum_{k=2}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) b_k$$

and defining

$$s_n = \zeta(n) - \frac{1}{n-1}$$

for $n \geq 2$, while leaving s_0 and s_1 temporarily indeterminate, one may invert the series as

$$b_n = s_0 - ns_1 + \sum_{k=2}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) \left[\zeta(k) - \frac{1}{k-1} \right]$$

The harmonic numbers appear as

$$\sum_{k=2}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{k-1} = 1 - n + nH_{n-1}$$

To complete the process, one need only to make the identification

$$b_0 = s_0 = \lim_{s \to 0} \zeta(s) - \frac{1}{s-1} = \zeta(0) + 1 = \frac{1}{2}$$

and

$$s_1 = \lim_{s \to 1} \zeta(s) - \frac{1}{s-1} = \gamma$$

Thus, the binomial transform as been demonstrated at the integer values of the zeta function.

Next, one must prove that the series

$$\frac{1}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{b_n}{n!} \left(\begin{array}{c} s \\ k \end{array} \right)$$

is identical to the Riemann zeta function for all complex values of s. To do this, one considers the function

$$f(s) = -\zeta(s) + \frac{1}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{b_n}{n!} \left(\begin{array}{c} s \\ k \end{array} \right)$$

and note that $f(s)$ vanishes at all non-negative integers. By Carlson's theorem (given below), $f(s)$ will vanish everywhere, provided that it can be shown that $f(s)$ is exponentially bounded in all directions for large $|s|$, and is exponentially bounded by $e^{c|s|}$for some real $c < \pi$ for s large in the imaginary direction.

XXX this proof should be supplied.

3. Generating Functions

The ordinary and the exponential generating functions for the coefficients b_n are readily obtained; these are presented in this section.

Theorem 3.1. *The ordinary generating function is given by*

$$\begin{aligned}\beta(z) &= \sum_{n=0}^{\infty} b_n z^n \\ &= \frac{-1}{2(1-z)} + \frac{z}{(1-z)^2}\left[\ln(1-z) + \psi\left(\frac{1}{1-z}\right)\right] \qquad (3.1)\end{aligned}$$

where ψ is the digamma function.

Proof. The above may be obtained by straightforward substitution followed by an exchange of the order of summation. The harmonic numbers lead to the logarithm via the identity

$$\ln(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$$

while the binomial series re-sums to the digamma function, which has the Taylor's series

$$\psi(1+y) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) y^{n-1}$$

(equation 6.3.14 from Abramowitz and Stegun[**?**]). □

The above simplifies considerably, to

$$\beta\left(\frac{w-1}{w}\right) = \frac{w}{2} + (w-1)(\psi(w) - \ln w)$$

upon substitution of $z = (w-1)/w$.

An exponential generating function is given by

$$\sum_{n=0}^{\infty} b_n \frac{z^n}{n!} = e^z \left[z\left(1-\gamma+\text{Ein}(z)\right) - \frac{1}{2} + \sum_{k=2}^{\infty} \frac{(-z)^k}{k!}\zeta(k)\right] \qquad (3.2)$$

where $\text{Ein}(z)$ is the entire exponential integral:

$$\text{Ein}(z) = -\sum_{k=1}^{\infty} \frac{1}{k}\frac{(-z)^k}{k!} = e^{-z}\sum_{n=1}^{\infty} H_n \frac{z^n}{n!} \qquad (3.3)$$

and H_n are the harmonic numbers.

Note. One is left to wonder what the function

$$\mu(s;x) = \frac{1}{s-1} + \sum_{n=0}^{\infty} (-)^n \left(\begin{array}{c} s \\ n \end{array}\right) b_n x^n \qquad (3.4)$$

might be like; it is presumably related to the Polylogarithm (Jonquiere's function) XXXX Do the scratching needed to get the relationship nailed down. Ugh.

4. Numerical Analysis

The series expression for the b_n is very conducive to numerical analysis. The values of the zeta function may be rapidly computed to extremely high precision using the Tchebysheff polynomial technique given by Borwein[**?**]. Thus, it becomes possible to explore the behavior of the b_n up to $n \sim 5000$ with only a moderate investment in coding and computer time. Note, however, that in order to get up to these ranges, computations must be performed keeping thousands of decimal places of precision, due to the very large values

TABLE 1. Some values of b_n

n	b_n
0	0.5
1	-0.07721566490153286060651209...
2	-0.00949726295483928474060901...
3	0.001098302680486442197971068...
4	0.001504029222654892992160356...
5	0.000715059124311572276195930...
6	0.000203986913616129918772188...
7	-0.000003382695485446052419661...
8	-0.00005422279825033492206785...

This table shows the first few values of b_n. These may be easily calculated to high precision if desired. As is immediately apparent, these get small quickly.

that the binomial coefficient can take. In the following, calculations were performed using the GNU Multiple Precision Arithmetic Library [**?**].

The table 1 shows the numerical values of the first few of these constants. It appears that the $b_n$are oscillatory, as shown in the figure 4.1. The values are also very rapidly decreasing, so that to the first order, that $|b_n| \sim \exp\left(-\sqrt{4\pi n}\right)$.

A numeric fit to the oscillatory behavior of the function may be made. An excellent fit for the $k'th$ zero is provided by

$$q(k) = \frac{\pi}{4}k^2 + \frac{9\pi}{16}k - \frac{1}{4k} + c + \frac{r}{k-a+ib} + \frac{r}{k-a-ib} \tag{4.1}$$

where $c = 0.44057997$, $2/r = 215.08$, $a = 3.19896$ and $b = 0.2898$. With these parameters, the fit to the zero-crossings for $k \geq 9$ is better than a part per million, while describing the lower crossings to a few parts per hundred. The leading quadratic term in this equation is easily invertible to give the oscillatory behavior of the b_n; the inversion is given by

$$s_n = \sin \pi \left(\frac{-9}{8} - \sqrt{\frac{81}{64} + 4\,\frac{(n + 0.44058)}{\pi}} \right) \tag{4.2}$$

XXX except this is a mediocre fit all by itself, and needs to be tweaked. To demonstrate the quality of this fit, the graph 4.2 shows a_n divided by s_n.

Asymptotically, the amplitude of the b_n appears to be given by

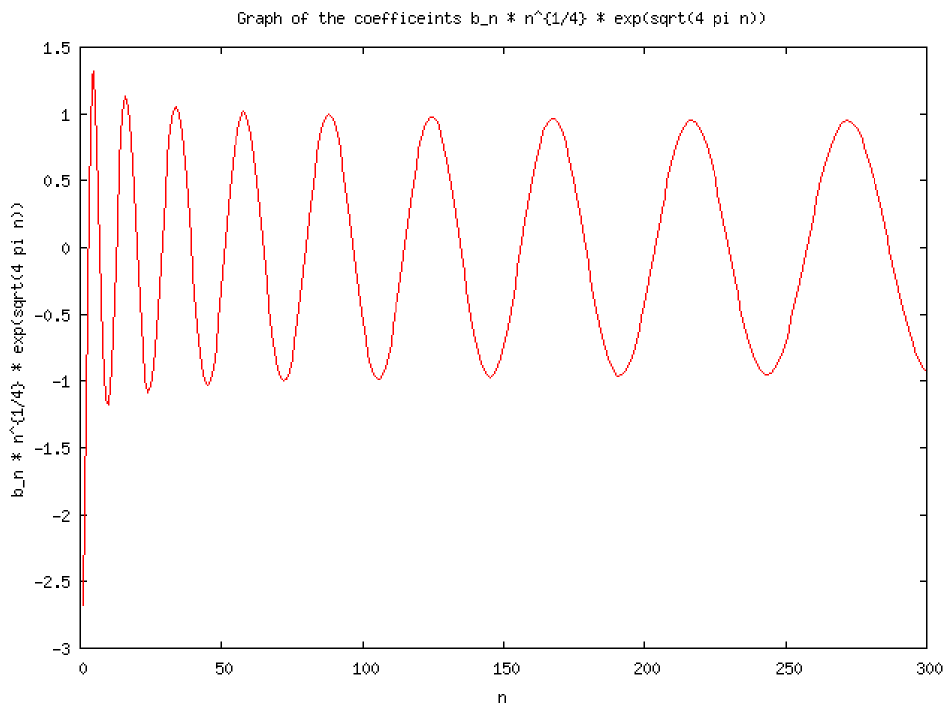
$$b_n \simeq s_n n^{1/4} \exp -\left(\sqrt{4\pi n}\right) \tag{4.3}$$

However, a more precise fit is strangely difficult. XXXX The below should be replaced or removed. An attempt to numerically fit the data with an expression of the form

$$a_n \simeq s_n \exp -\left(b + (c + dn)^{\epsilon}\right) \tag{4.4}$$

results in fit parameters of $b = 3.9 \pm 1.5$ and $c = -50 \pm 100$ and $d = 12.87 \pm 0.15$ and $\epsilon = 0.5 \pm 0.007$. This fit is not very pleasing; in particular, the true asymptotic behavior doesn't seem to set in until n is larger than several hundred, which is unexpectedly high.

FIGURE 4.1. Graph of the first few values of b_n



This figure shows a graph of the first 300 values of b_n normalized by a factor of $n^{-1/4}\exp\sqrt{4\pi n}$. Oscillation is clearly visible; the period of oscillation is slowly increasing. The figure also makes clear that the asymptotic behavior is bounded.

5. EXTENSION OF b_n TO COMPLEX ARGUMENTS

The $b_n$may be generalized by taking the parameter n to be a complex number. The "obvious" and reasonable generalization is

$$b(s)=-s\gamma+\frac{1}{2}+\sum_{k=2}^{\infty}(-1)^k\left(\begin{array}{c}s\\k\end{array}\right)\left[\zeta(k)-\frac{1}{k-1}\right] \tag{5.1}$$

By construction, the function $b(s)$ interpolates the previous series, so that $b(n)=b_n$ for non-negative integers n. This function has a pole at $s=-1$, and the above series expansion is convergent only for $\Re s>-1$.

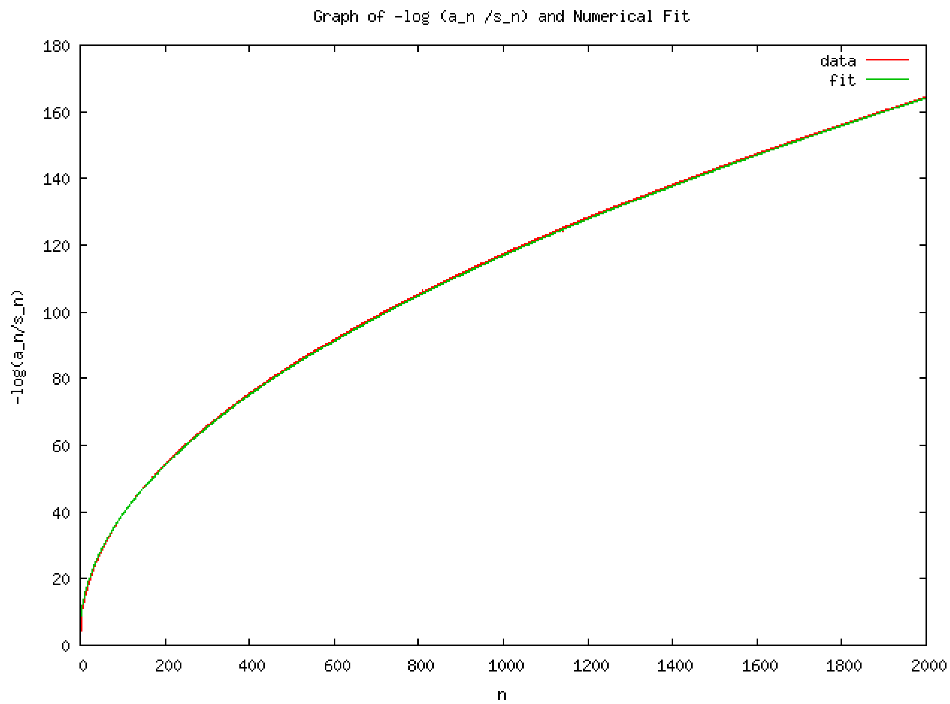
One may ask what form the analytic continuation of $b(s)$ to the entire complex plane may look like. To examine this question, one makes use of two identities.

First, one has

$$\sum_{k=0}^{\infty}(-x)^k\left(\begin{array}{c}s\\k\end{array}\right)=(1-x)^s$$

which holds for $|x|<1$ for any s. When s is a positive integer, the sum on the left clearly vanishes in the limit of $x\to1$, because the sum consists of a finite number of terms. That it also vanishes for any s with $\Re s>0$, follows from Carlson's theorem. The theorem applies

FIGURE 4.2. Graph of the log of the amplitude a_n/s_n



This figure shows a graph of $-\log(a_n/s_n)$ out through $n \sim 2000$. It is essentially a graph of the log of the amplitude of the oscillations of a_n. Note the absence of any 'blips' that would occur if the sinusoidal fit to the data was poor; the sinusoidal curve s_n models the data remarkably well. This figure also illustrates a fit to the amplitude. Specifically, the fit curve is given by $3 + 3.6\sqrt{n+1}$. Note that this fit fails badly for values of $n \lesssim 20$ and that for larger n, the fit is only good to $3 + 3.6\sqrt{n+1} + \log(a_n/s_n) \simeq 0 \pm 0.5$ on average.

Xxx-this graph should be replaced or removed.

because, for any s, one can has that $|s\log(1-x)| < \pi$ in the limit of $x \to 1$. Thus, for any s with $\Re s > 0$, one has

$$\sum_{k=2}^{\infty}(-1)^k \left(\begin{array}{c} s \\ k \end{array} \right) = s - 1$$

and one may take the right hand side to be the analytic continuation, to the entire complex plane, of the left hand side.

Another important identity uses the digamma function

$$\psi(s) = \frac{d}{ds} \ln \Gamma(s)$$

to generalize the harmonic numbers

$$\psi(n) = H_{n-1} - \gamma$$

The digamma has a suitable binomial series [XXX ref for this formula?] valid for $s>-1$:

$$\psi(s+1)=-\gamma-\sum_{k=1}^{\infty}(-1)^{k}\left(\begin{array}{c} s\\ k \end{array}\right)\frac{1}{k}$$

As before, the uniqueness of this expansion may be argued by appeals to Carlson's theorem. Substituting

$$\frac{1}{k}=\frac{1}{k+1}+\frac{1}{k(k+1)}$$

and making use of the previous identity, one can readily show that, for $s>0$,

$$\sum_{k=2}^{\infty}(-1)^{k}\left(\begin{array}{c} s\\ k \end{array}\right)\frac{1}{k-1}=1+s\left(\gamma-1+\psi(s)\right)$$

Combining the above, one may write $b(s)$ in a form that is regular on the entire complex plane

$$b(s)=-\frac{3}{2}+s\left[2\left(1-\gamma\right)-\psi(s)\right]+\sum_{k=2}^{\infty}(-1)^{k}\left(\begin{array}{c} s\\ k \end{array}\right)\left[\zeta(k)-1\right]$$

The sum on the left is entire, having no poles except at infinity, and converges for all complex values of s. That it is entire follows easily, as the sum is bounded:

$$\begin{aligned} \left|\sum_{k=2}^{\infty}(-1)^{k}\left(\begin{array}{c} s\\ k \end{array}\right)\left[\zeta(k)-1\right]\right| &< 2\left|\sum_{k=2}^{\infty}\left(-\frac{1}{2}\right)^{k}\left(\begin{array}{c} s\\ k \end{array}\right)\left[\zeta(k)-1\right]\right| \\ &\le \left|2^{1-s}-2+s\right| \end{aligned}$$

and thus, the sum cannot have any poles for finite s. The digamma is regular and single-valued on the complex plane, having simple poles at the negative integers; and thus, $b(s)$ is likewise.

XXXX remove or replace The figure 5.1 illustrates $a(s)$ on the imaginary axis; the graphic 5.2 illustrates the phase of $a(s)$ in the upper-right complex plane. A numerical exploration of this function seems to indicate that the only zeros of this function occur on the real number line.

6. Relation to the Stieltjes constants

The standard power series for the Riemann zeta function is written as

$$\zeta(s)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\gamma_{n}(s-1)^{n} \tag{6.1}$$

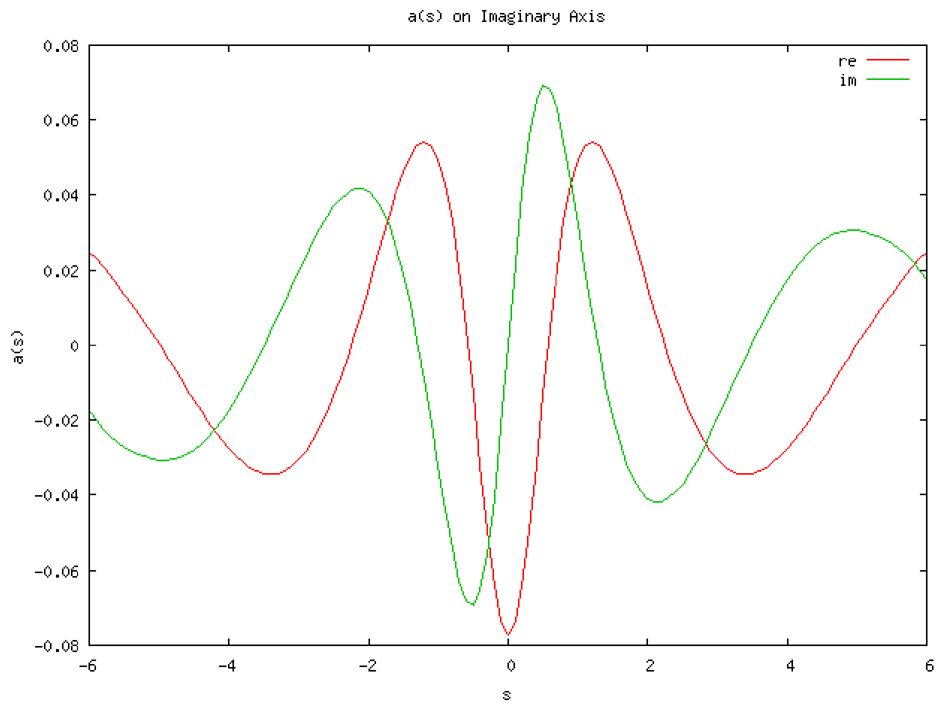
where the coefficients γ_n are known as the Stieltjes constants. The b_n can be related to these by re-expressing the binomial coefficient as a power series. That is, the binomial coefficient is just a polynomial

$$\left(\begin{array}{c} s\\ n \end{array}\right)=\frac{(s)_{n}}{n!}=\frac{(-1)^{n}}{n!}\sum_{k=0}^{n}(-1)^{k}\left[\begin{array}{c} n\\ k \end{array}\right]s^{k} \tag{6.2}$$

with coefficients given by $\left[\begin{array}{c} n\\ k \end{array}\right]$, the Stirling Number of the First Kind. The sign convention used here is that the Stirling numbers are given by the recursion relation

$$\left[\begin{array}{c} n+1\\ k \end{array}\right]=n\left[\begin{array}{c} n\\ k \end{array}\right]+\left[\begin{array}{c} n\\ k-1 \end{array}\right]$$

FIGURE 5.1. $a(s)$ on the Imaginary Axis



This figure shows the real and imaginary parts of $a(s)$ along the imaginary axis. The function appears to be oscillatory. Note that the series expansion converges only with great difficulty as one goes further out on the imaginary axis. XXX top be replaced by phase plot of b(s), which should look similar.

making them all positive, and

$$\left[\begin{array}{c} n \\ 0 \end{array}\right] = \delta_{n,0}$$

This convention differs by a sign from the other common convention for the Stirling numbers $s(n,k)$, as

$$\left[\begin{array}{c} n \\ k \end{array}\right] = (-1)^{n-k} s(n,k)$$

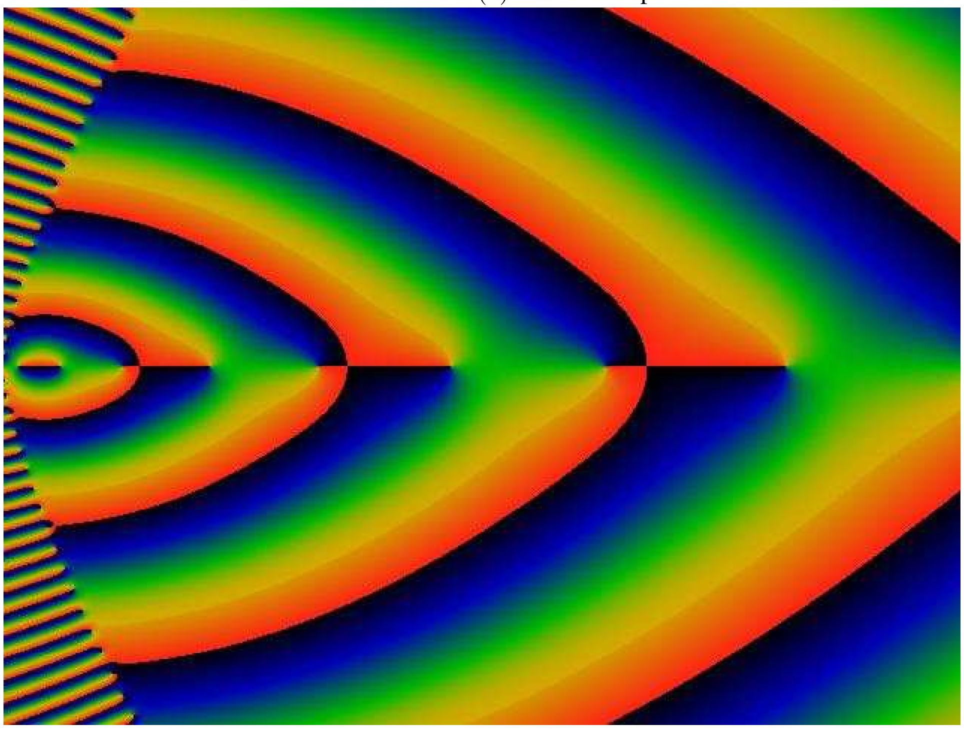
Straightforward substitution leads to

$$\frac{(-1)^m}{m!}\gamma_m = \sum_{n=m}^{\infty} \frac{b_n}{n!} \sum_{k=m}^{n} (-1)^k \left[\begin{array}{c} n \\ k \end{array}\right] \left(\begin{array}{c} k \\ m \end{array}\right) \tag{6.3}$$

The operator

$$G_{nm} = (-1)^{m+1} \sum_{k=m}^{n} (-1)^k \left[\begin{array}{c} n \\ k \end{array}\right] \left(\begin{array}{c} k \\ m \end{array}\right) \tag{6.4}$$

FIGURE 5.2. Phase of $a(s)$ on the Complex Plane



This figure shows the phase of $a(s)$ on the right half of the complex plane, running along the real-axis from -2 to +62 and the imaginary axis from -24 to about +24. The ribbing on the left is entirely due to numerical errors, as the sums converge only with great difficulty. The color scheme is such that black corresponds to $-\pi$, green to 0 and red to $+\pi$, on a smooth scale. Thus, the red-to-black color discontinuities, correspond to a phase change of 2π. Note that these terminate on zeros or poles; in this case, it appears that there is only one pole, at $s = -1$, and all others are zeros. In particular, it appears that all of the zeros occur on the real number line.

XXX to be replaced by plot of b(s), which should look similar.

is lower-triangular. There is an even simpler, amusing identity which may be obtained by exchanging the order of integration:

$$\frac{(-1)^m}{m!}\gamma_m = \sum_{k=m}^{\infty}(-1)^k \left(\begin{array}{c} k \\ m \end{array} \right) \sum_{n=k}^{\infty} \frac{b_n}{n!} \left[\begin{array}{c} n \\ k \end{array} \right] \tag{6.5}$$

The first sum sum be recognized as the so-called binomial transform $\{s_m\}$ of a series $\{a_k\}$

$$s_m = \sum_{k=m}^{\infty}(-1)^k \left(\begin{array}{c} k \\ m \end{array} \right) a_k$$

Such a transform is an involution, so that its inverse is given by

$$a_n = \sum_{m=n}^{\infty}(-1)^m \left(\begin{array}{c} m \\ n \end{array} \right) s_m$$

Applying this to the above leads to the amusing identity

$$\frac{c_k}{k!} = \sum_{m=k}^{\infty} \frac{\gamma_m}{m!} \left(\begin{array}{c} m \\ k \end{array} \right) = \sum_{n=k}^{\infty} \frac{b_n}{n!} \left[\begin{array}{c} n \\ k \end{array} \right]$$

where the constants c_k correspond to derivatives of the zeta function at $s = 0$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} c_k \frac{(-s)^k}{k!}$$

The relationship between the constants b_n and the Stieltjes constants provide a possible avenue for exploring the asymptotic behavior of the Stieltjes constants. This is explored in the next section.

7. Asymptotic Behavior of the Stieltjes Constants

Perhaps one can capture the asymptotic behavior of the Stieltjes constants by understanding the asymptotic behavior of the operator G_{nm}. That this might be possible is suggested by the relation

$$\frac{\gamma_m}{m!} = -\sum_{n=m}^{\infty} \frac{b_n}{n!} G_{nm} \tag{7.1}$$

with G_{nm} defined as in equation 6.4. Unfortunately, this proves rather hard. The rest of this section explores the asymptotic behavior of G_{nm}.

The table below shows some of the low values of this operator:

G_{nm}	0	1	2	3	4	5	6	7	8	9
0	-1									
1	1	-1								
2	0	1	-1							
3	0	1	0	-1						
4	0	2	1	-2	-1					
5	0	6	5	-5	-5	-1				
6	0	24	26	-15	-25	-9	-1			
7	0	120	154	-49	-140	-70	-14	-1		
8	0	720	1044	-140	-889	-560	-154	-20	-1	
9	0	5040	8028	64	-6363	-4809	-1638	-294	-27	-1

Several interesting patterns are clearly visible in this table. Most of these follow easily from the recurrence relationship for the entries in this table. One has that

$$G_{nm} = (n-2)G_{n,m-1} + G_{n-1,m-1} \tag{7.2}$$

which is easily proven by direct substitution and the application of the recurrence relations for the Stirling numbers and binomial coefficients. The zeroth column follows from a well-known relationship on the Stirling numbers:

$$G_{n,0} = -\sum_{k=0}^{n} (-1)^k \left[\begin{array}{c} n \\ k \end{array} \right] = \delta_{n,1} - \delta_{n,0} \tag{7.3}$$

The next column follows readily from the recursion relation 7.2; one has

$$G_{n,1} = (n-2)! \tag{7.4}$$

With only a bit more work, one finds that

$$G_{n,2} = (n-2)!\,(H_{n-2} - 1) \tag{7.5}$$

with H_n the harmonic numbers, as before. Asymptotically, for large n, one has

$$H_n = \log n - \gamma + \frac{1}{2n} + \mathcal{O}\left(\frac{1}{12n^2}\right)$$

The general asymptotic pattern that ensues may be better seen by defining

$$g_{nm} = \frac{G_{nm}}{(n-2)!}$$

so that one has

$$g_{n,2} = \log n + \mathcal{O}(1)$$

for large n. The next column has the recursion relation

$$g_{n,3} = g_{n-1,3} + \frac{H_{n-3} - 1}{n-2}$$

which is easily solvable as

$$g_{n,3} = \sum_{k=1}^{n-2} \frac{H_{k-1} - 1}{k} = -1 - H_{n-2} + \sum_{k=1}^{n-3} \frac{H_k}{k+1} \tag{7.6}$$

Although initially negative, the $g_{n,3}$ eventually turn positive at $n = 9$. This general pattern, of entries in a column starting negative and eventually turning positive, seems to be the asymptotic behavior. Another representative case is shown in the graph 7.1.

The general pattern of the last row of the table appears to persist asymptotically: the first few entries are positive, then swing sharply negative before trailing off to small values. This is illustrated in the graphic 7.2.

The upshot of this exercise indicates that the asymptotic behavior of $g_{n,m}$ is difficult to ascertain. Even if an expression were in hand, it would then have to be integrated with the asymptotic behavior for the b_n before the asymptotic behavior for the Stieltjes constants were discernible.

8. Off-Topic

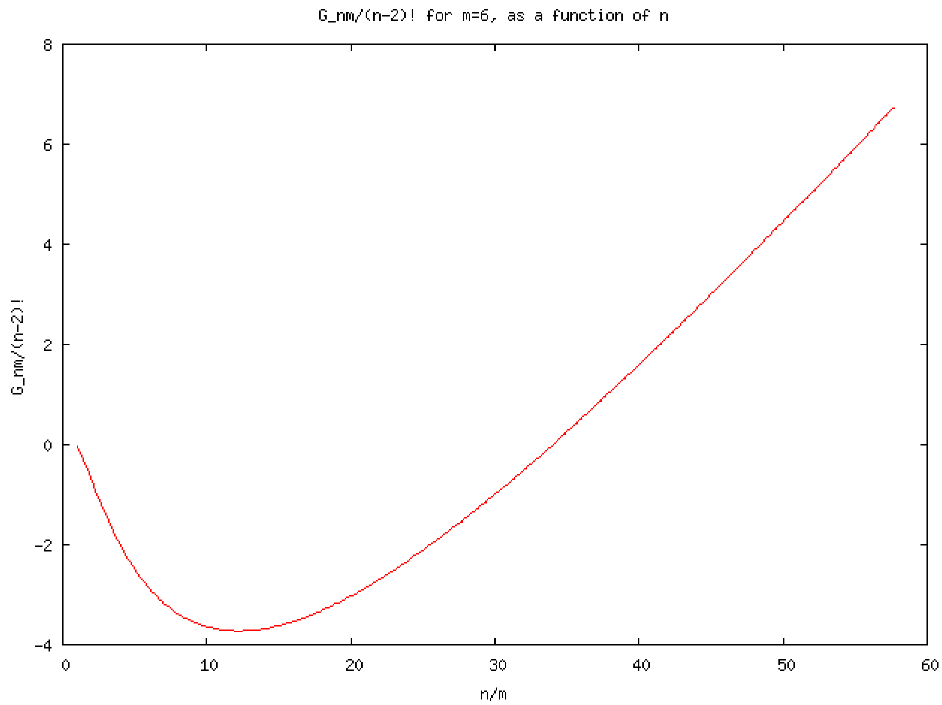
Note. The following is off-topic, not sure what to do with it.

While on the topic of Umbral relations, the application of Newton's divided differences to the Riemann zeta function leads to the curious function

$$Q(z) = \sum_{n=0}^{\infty} \binom{n+1}{z-1} \left[\zeta(n+2) - 1\right] \tag{8.1}$$

which has the curious properties that $Q(n) = \zeta(n)$ for all integers $n \geq 2$. The pole is absent: $Q(1) = 1$ and $Q(n) = 0$ $\forall$integers $n \leq 0$. The analytic structure of $Q(z)$ is unclear.xxx Bessel function!?!

FIGURE 7.1. $g_{n,m}$ as a function of n for fixed $m = 6$.



This figure shows $g_{n,m} = G_{n,m}/(n-2)!$ as a function of n, at m fixed at $m = 6$. Figures for larger values of m follow the same general shape, with the exception that the negative dip is entered more slowly, is far deeper, and lasts considerably longer. Conjecturally, the depth and length of the negative dip appears to be of order $\Gamma(m)$.

9. FINITE DIFFERENCES AND THE NORLUND-RICE INTEGRAL

This section reviews the the definition of the Norlund-Rice integral, and its application to finite differences and Newton series. Theorems required for convergence, and in particular, Carlson's theorem, are briefly stated. The next section applies these techniques to the evaluation of the finite differences of the Riemann zeta function.

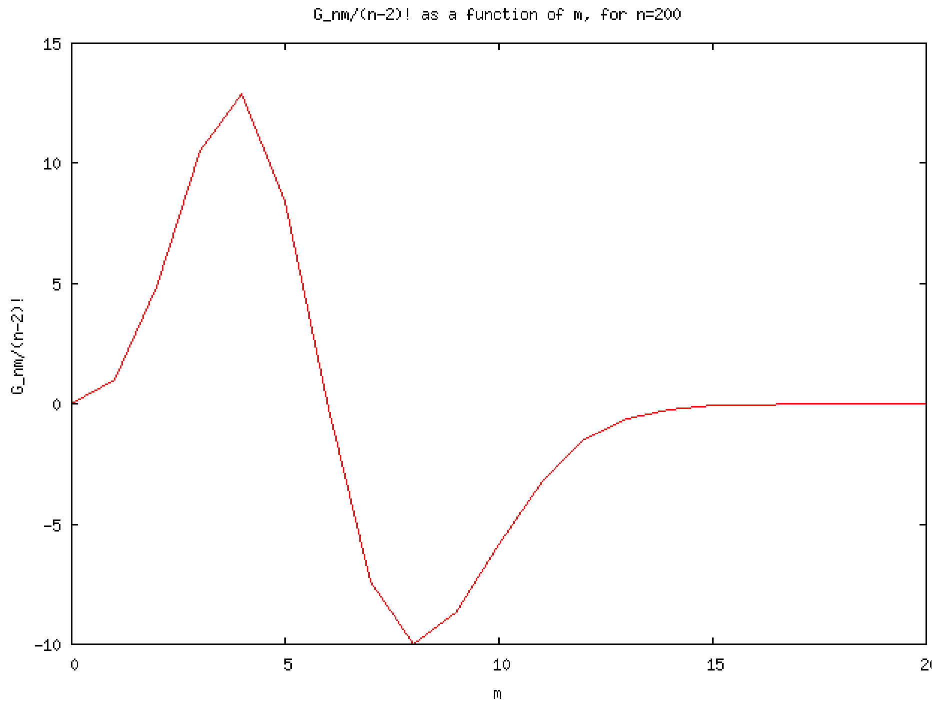
Lemma 9.1. *(Norlund-Rice ?)*
Given a function $f(s)$ holomorphic on a region containing the integers $\{n_0, \cdots, n\}$, then the finite differences of the sequence $\{f(k)\}$ may be given an integral representation

$$\sum_{k=n_0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (-1)^k f(k) = \frac{(-1)^n}{2\pi i} \oint_{\mathcal{C}} f(s) \frac{n!}{s(s-1)\cdots(s-n)} ds$$

where the contour of integration $\mathcal{C}$encircles the integers $\{n_0, \cdots, n\}$ in a positive direction, but does not include any of the integers $\{0, 1, \cdots, n_0 - 1\}$.

Proof. By residues.[**?**] $\square$

FIGURE 7.2. $g_{nm}$for $n = 200$



This figure shows $g_{nm} = G_{nm}/(n-2)!$ as a function of m, for $n = 200$. This figure is representative; figures for both larger and smaller n show the same general pattern: the first few coefficients are positive, the next few are negative, in a more-or-less smoothly varying fashion, decaying exponentially to zero for large m. The zero crossing shifts rightwards very slowly for increasing n; conjecturally, it goes presumably as $\Gamma^{-1}(n)$. Noteworthy seems the be the fact that the positive and negative swings are approximately equal in magnitude.

Such sums commonly occur in the theory of finite differences. Given a function $f(x)$, one defines its forward differences at point $x = a$ by

$$\Delta^n [f] (a) = \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} f(p+a) \tag{9.1}$$

where $\binom{n}{p}$ is the binomial coefficient. When one has only an arithmetic function or sequence of values $f_p = f(p)$, rather than a function of a continuous variable x, the above is referred to as the binomial transform of the sequence.

The forward differences may be used to construct the umbral calculus analog of a Taylor's series:

$$g(z+a) = \sum_{n=0}^{\infty} \Delta^n [f] (a) \frac{(z)_n}{n!} \tag{9.2}$$

where $(z)_n = z(z-1)\dots(z-n+1)$ is the Pochhammer symbol or falling factorial. When the growth rate of f is suitably limited on the complex plane, then, by means of Carlson's Uniqueness theorem, $g = f$, and this series is known as the Newton series for f.

Theorem 9.2. *(Carlson, 1914) If a function $f(z)$ vanishes on the positive integers, and if it is of exponential type, that is, if*

$$f(z) = \mathcal{O}(1)e^{\tau|z|}$$

for $z \in \mathbb{C}$ and for some real $\tau < \infty$, and if furthermore its growth is bounded on the imaginary axis, so that

$$f(iy) = \mathcal{O}(1)e^{-c|y|}$$

for $c < \pi$, then f is identically zero.

Proof. See [**?**]. [XXX this ref doesn't actually contain proof] □

An example of a function that violates Carlson's theorem is $\sin \pi z$, which vanishes on the integers, but has a growth rate of $c = \pi$ along the imaginary axis.

10. Asymptotic analysis

This section describes the Norlund-Rice asymptotic analysis for the Riemann b_n. Should be cut-n-paste from Flajolet's early email ... To Do xxx.

11. Dirichlet L-functions

The following section performs the asymptotic analysis for the Newton series of the Dirichlet L-functions; this section briefly reviews their definition. The Dirichlet L-functions[**?**] are defined in terms of the Dirichlet characters, which are group representation characters of the cyclic group. They play an important role in number theory, and the Riemann hypothesis generalizes to the L-functions. The Dirichlet characters are multiplicative functions, and are periodic modulo k. That is, a character $\chi(n)$ is an arithmetic function of an integer n, with period k, such that $\chi(n+k) = \chi(n)$. A character is multiplicative, in that $\chi(mn) = \chi(m)\chi(n)$ for all integers m, n. Furthermore, one has that $\chi(1) = 1$ and $\chi(n) = 0$ whenever $\gcd(n,k) \neq 1$. The L-function associated with the character $\chi$is defined as

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{11.1}$$

All such L-functions may be re-expressed in terms of the Hurwitz zeta function as

$$L(\chi, s) = \frac{1}{k^s} \sum_{m=1}^{k} \chi(m) \zeta\left(s, \frac{m}{k}\right) \tag{11.2}$$

where k is the period of χ and $\zeta(s,q)$ is the Hurwitz zeta function, given by

$$\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s} \tag{11.3}$$

Thus, the study of the analytic properties of the L-functions can be partially unified through the study of the Hurwitz zeta function.

12. Forward differences

In analogy to the study of the forward differences of the Riemann zeta function, the remainder of this paper will concern itself with the analysis of the series given by

$$L_n = \sum_{p=2}^{n} (-1)^p \left(\begin{array}{c} n \\ p \end{array} \right) L(\chi, p) \tag{12.1}$$

Because of the relation 11.2 connecting the Hurwitz zeta function to the L-function, it is sufficient to study sums of the form

$$A_n(m,k) = \sum_{p=2}^{n} (-1)^p \left(\begin{array}{c} n \\ p \end{array} \right) \frac{\zeta\left(p, \frac{m}{k}\right)}{k^p} \tag{12.2}$$

since

$$L_n = \sum_{m=1}^{k} \chi(m)\, A_n(m,k) \tag{12.3}$$

Converting the sum to the Norlund-Rice integral, and extending the contour to the half-circle at positive infinity, and noting that the half-circle does not contribute to the integral, one obtains

$$A_n(m,k) = \frac{(-1)^n}{2\pi i}\, n! \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\zeta\left(s, \frac{m}{k}\right)}{k^s s(s-1)\cdots(s-n)}\, ds \tag{12.4}$$

Moving the integral to the left, one encounters single pole at $s=0$ and a double pole at $s=1$. The residue of the pole at $s=0$ is

$$\text{Res}(s=0) = \zeta\left(0, \frac{m}{k}\right) \tag{12.5}$$

where one has the curious identity in the form of a multiplication theorem for the digamma function:

$$\zeta\left(0, \frac{m}{k}\right) = \frac{-1}{\pi k} \sum_{p=1}^{k} \sin\left(\frac{2\pi p m}{k}\right) \psi\left(\frac{p}{k}\right) = -B_1\left(\frac{m}{k}\right) = \frac{1}{2} - \frac{m}{k} \tag{12.6}$$

Here, ψ is the digamma function and $B_1$is the Bernoulli polynomial of order 1. The double pole at $s=1$ evaluates to

$$\text{Res}(s=1) = \frac{n}{k}\left[\psi\left(\frac{m}{k}\right) + \ln k + 1 - H_{n-1}\right] \tag{12.7}$$

Combining these, one obtains

$$A_n(m,k) = \left(\frac{m}{k} - \frac{1}{2}\right) - \frac{n}{k}\left[\psi\left(\frac{m}{k}\right) + \ln k + 1 - H_{n-1}\right] + a_n(m,k) \tag{12.8}$$

The remaining term has the remarkable property of being exponentially small; that is,

$$a_n(m,k) = \mathcal{O}\left(e^{-\sqrt{Kn}}\right) \tag{12.9}$$

for a constant K of order m/k. The next section develops an explicit asymptotic form for this term.

13. Saddle-point methods

The term $a_n(m,k)$ is represented by the integral

$$a_n(m,k) = \frac{(-1)^n}{2\pi i}\, n! \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\zeta\left(s, \frac{m}{k}\right)}{k^s s(s-1)\cdots(s-n)}\, ds \tag{13.1}$$

which resulted from shifting the integration contour past the poles. At this point, the functional equation for the Hurwitz zeta may be applied. This equation is

$$\zeta\left(1-s, \frac{m}{k}\right) = \frac{2\Gamma(s)}{(2\pi k)^s} \sum_{p=1}^{k} \cos\left(\frac{\pi s}{2} - \frac{2\pi pm}{k}\right) \zeta\left(s, \frac{p}{k}\right) \tag{13.2}$$

This allows the integral to be expressed as
(13.3)

$$a_n(m,k) = -\frac{n!}{k\pi i} \sum_{p=1}^{k} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{(2\pi)^s} \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)} \cos\left(\frac{\pi s}{2} - \frac{2\pi pm}{k}\right) \zeta\left(s, \frac{p}{k}\right) ds$$

It will prove to be convenient to pull the phase factor out of the cosine part; we do this now, and write this integral as

$$\begin{aligned} a_n(m,k) &= -\frac{n!}{2k\pi i} \sum_{p=1}^{k} \exp\left(i\frac{2\pi pm}{k}\right) \times \\ &\qquad \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{(2\pi)^s} \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)} \exp\left(-i\frac{\pi s}{2}\right) \zeta\left(s, \frac{p}{k}\right) ds \\ &\quad + c.c. \end{aligned} \tag{13.4}$$

where $c.c.$ means that i should be replaced by $-i$ in the two exp parts.

For large values of n, this integral may be evaluated by means of the saddle-point method. The saddle-point method, or method of steepest descents, may be applied whenever the integrand can be approximated by a sharply peaked Gaussian, as the above can be for large n. More precisely, The saddle-point theorem states that

$$\int e^{-Nf(x)} dx \approx \sqrt{\frac{2\pi}{N\left|f''(x_0)\right|}}\, e^{-Nf(x_0)} \left[1 - \frac{f^{(4)}(x_0)}{8N\left|f''(x_0)\right|^2} + \cdots\right] \tag{13.5}$$

is an asymptotic expansion for large N. Here, the function f is taken to have a local minimum at $x = x_0$ and $f''(x_0)$ and $f^{(4)}(x_0)$ are the second and fourth derivatives at the local minimum.

To recast the equation 13.4 into the form needed for the method of steepest descents, an asymptotic expansion of the integrands will need to be made for large n. After such an expansion, it is seen that the saddle point occurs at large values of s, and so an asymptotic expansion in large s is warranted as well. As it is confusing and laborious to simultaneously expand in two parameters, it is better to seek out an order parameter to couple the two. This may be done as follows. One notes that the integrands have a minimum, on the real s axis, near $s = \sigma_0 = \sqrt{\pi pn/k}$ and so the appropriate scaling parameter is $z = s/\sqrt{n}$. One should then immediately perform a change of variable from s to z. The asymptotic expansion is then performed by holding z constant, and taking n large. Thus, one writes

$$a_n(m,k) = -\frac{1}{2k\pi i}\sum_{p=1}^{k}\left[\exp\left(i\frac{2\pi pm}{k}\right)\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{f(z)}dz + \exp\left(-i\frac{2\pi pm}{k}\right)\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{\overline{f}(z)}dz\right] \tag{13.6}$$

where $\overline{f}$ is the complex conjugate of f.

Proceeding, one has

$$f(z) = \log n! + \frac{1}{2}\log n + \phi\left(z\sqrt{n}\right) \tag{13.7}$$

and

$$\phi(s) = -s\log\left(\frac{2\pi p}{k}\right) - i\frac{\pi s}{2} + \log\frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)} + \mathcal{O}\left(\left(\frac{p}{k+p}\right)^s\right) \tag{13.8}$$

where the approximation that $\zeta(s,p/k) = (k/p)^s + \mathcal{O}\left((p/(k+p))^s\right)$ for large s has been made. More generally, one has

$$\log\zeta(s) = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\log n} \tag{13.9}$$

where $\Lambda(n)$ is the von Mangoldt function. (XXX What about Hurwitz?) The asymptotic expansion for the Gamma function is given by the Stirling expansion,

$$\log\Gamma(x) = \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log 2\pi + \sum_{j=1}^{\infty}\frac{B_{2j}}{2j(2j-1)x^{2j-1}} \tag{13.10}$$

and B_k are the Bernoulli numbers. Expanding to $\mathcal{O}(1/n)$ and collecting terms, one obtains

$$\begin{aligned} f(z) &= \frac{1}{2}\log n - z\sqrt{n}\left[\log\frac{2\pi p}{k} + i\frac{\pi}{2} + 2 - 2\log z\right] \\ &\quad + \log 2\pi - 2\log z - \frac{z^2}{2} \\ &\quad + \frac{1}{6z\sqrt{n}}\left[10+z^2\right] + \frac{1}{2n}\left[1 - \frac{z^2}{2} - \frac{z^4}{6} + \frac{73}{72z^2}\right] + \mathcal{O}\left(n^{-3/2}\right) \end{aligned} \tag{13.11}$$

The saddle point may be obtained by solving $f'(z) = 0$. To lowest order, one obtains $z_0 = (1+i)\sqrt{\pi p/k}$. To use the saddle-point formula, one needs $f''(z) = 2\sqrt{n}/z + \mathcal{O}(1)$. Substituting, one directly obtains

$$\begin{aligned} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{f(z)}dz &= \left(\frac{2\pi^3 pn}{k}\right)^{1/4} e^{i\pi/8}\exp\left(-(1+i)\sqrt{\frac{4\pi pn}{k}}\right) \\ &\quad + \mathcal{O}\left(n^{-1/4}e^{-2\sqrt{\pi pn/k}}\right) \end{aligned} \tag{13.12}$$

while the integral for $\overline{f}$ is the complex conjugate of this (having a saddle point at the complex conjugate location). Inserting this into equation 13.6 gives

$$
\begin{aligned}
a_n(m,k) &= \frac{1}{k}\left(\frac{2n}{\pi}\right)^{1/4}\sum_{p=1}^{k}\left(\frac{p}{k}\right)^{1/4}\exp\left(-\sqrt{\frac{4\pi pn}{k}}\right)\times \\
&\qquad \cos\left(\sqrt{\frac{4\pi pn}{k}}-\frac{5\pi}{8}-\frac{2\pi pm}{k}\right) \\
&\quad +\mathcal{O}\left(n^{-1/4}e^{-2\sqrt{\pi pn/k}}\right)
\end{aligned}
\tag{13.13}
$$

For large n, only the $p=1$ term contributes significantly, and so one may write

$$
\begin{aligned}
a_n(m,k) &= \frac{1}{k}\left(\frac{2n}{\pi k}\right)^{1/4}\exp\left(-\sqrt{\frac{4\pi n}{k}}\right)\times \\
&\qquad \cos\left(\sqrt{\frac{4\pi n}{k}}-\frac{5\pi}{8}-\frac{2\pi m}{k}\right) \\
&\quad +\mathcal{O}\left(n^{-1/4}e^{-2\sqrt{\pi n/k}}\right)
\end{aligned}
\tag{13.14}
$$

which demonstrates the desired result: the terms a_n are exponentially small.

The consistency of these results, with respect to the previous derivation for the Riemann zeta can be checked inn several ways. First, one may make the direct substitution $m = k = 1$ to obtain

$$
\begin{aligned}
b_n &= a_n(1,1) = \left(\frac{2n}{\pi}\right)^{1/4}\exp\left(-\sqrt{4\pi n}\right)\cos\left(\sqrt{4\pi n}-\frac{5\pi}{8}\right) \\
&\quad +\mathcal{O}\left(n^{-1/4}e^{-2\sqrt{\pi n}}\right)
\end{aligned}
\tag{13.15}
$$

Alternately, the so-called "multiplication theorem" for the Hurwitz zeta states that

$$\sum_{m=1}^{k}\zeta\left(s,\frac{m}{k}\right)=k^s\zeta(s) \tag{13.16}$$

from which one may deduce both that

$$\sum_{k=1}^{m}A_n(m,k)=\sum_{p=2}^{n}(-1)^p\binom{n}{p}\zeta(p) \tag{13.17}$$

and that

$$\sum_{k=1}^{m}a_n(m,k)=b_n \tag{13.18}$$

In particular, the above must hold order by order in the asymptotic expansion. The correctness of the expansion given by equation 13.13 with regards to this identity may be readily verified by substitution. An important special case of equation 13.18 is the relation

$$a_n(2,2)=b_n-a_n(1,2)=\sum_{p=2}^{n}(-1)^p\binom{n}{p}\left(1-2^{-p}\right)\zeta(p) \tag{13.19}$$

which appears often in the literature [Coffey]. It has the asymptotic expansion

$$a_n(2,2) = \frac{1}{2}\sum_{p=1}^{2}\left(\frac{np}{\pi}\right)^{1/4}\exp\left(-\sqrt{2\pi np}\right)\cos\left(\sqrt{2\pi np}-\frac{5\pi}{8}\right) + \mathcal{O}\left(n^{-1/4}e^{-\sqrt{2\pi n}}\right) \tag{13.20}$$

XXX Coffey's sum is actually $A_n(1,2)$!!?? viz the Dirichlet eta ??

14. More L-functions

We conclude by briefly returning to the structure of the Dirichlet L-functions. The L-function coefficients defined in equation 12.1 are now given by

$$L_n = \sum_{m=1}^{k}\chi(m)A_n(m,k) \tag{14.1}$$

Writing

$$A_n = B_n + a_n \tag{14.2}$$

so that $B_n(m,k)$represents the non-exponential part, one may state a few results. For the non-principal characters, one has $\sum_{m=1}^{k}\chi(m)=0$ and thus, the first term simplifies to

$$\sum_{m=1}^{k}\chi(m)B_n(m,k) = \frac{1}{k}\sum_{m=1}^{k}\chi(m)\left[m - n\psi\left(\frac{m}{k}\right)\right] \tag{14.3}$$

For the principal character χ_1, one has $\sum_{m=1}^{k}\chi_1(m)=\varphi(k)$ with $\varphi(k)$ the Euler Totient function. Thus, for the principal character, one obtains

$$\sum_{m=1}^{k}\chi_1(m)B_n(m,k) = -\varphi(k)\left[\frac{1}{2}+\frac{n}{k}\left(\ln k + 1 - H_{n-1}\right)\right] + \frac{1}{k}\sum_{m=1}^{k}\chi_1(m)\left[m - n\psi\left(\frac{m}{k}\right)\right] \tag{14.4}$$

By contrast, the exponentially small term invokes a linear combination of Gauss sums. The Gauss sum associated with a character χ is

$$G(n,\chi) = \sum_{m\ \mathrm{mod} k}\chi(m)e^{2\pi imn/k} \tag{14.5}$$

and so, to leading order

$$\begin{aligned}\sum_{m=1}^{k}\chi(m)a_n(m,k) &= \frac{1}{2k}\sum_{p=1}^{k}\left(\frac{2pn}{\pi k}\right)^{1/4}\exp\left(-\sqrt{\frac{4\pi pn}{k}}\right)\times \\ &\quad \left[\exp i\left(\frac{5\pi}{8}-\sqrt{\frac{4\pi pn}{k}}\right)G(p,\chi) + \exp -i\left(\frac{5\pi}{8}-\sqrt{\frac{4\pi pn}{k}}\right)G(-p,\chi)\right] \\ &\quad +\mathcal{O}\left(n^{-1/4}e^{-2\sqrt{\pi n/k}}\right)\end{aligned} \tag{14.6}$$

The above expression simplifies slightly for the principle character, since one has the identities

$$G(1,\chi_1) = \mu(k) \tag{14.7}$$

with $\mu(k)$ the Mobius function and more generally,

$$G(p,\chi_1) = \frac{\varphi(k)\mu\left(\frac{k}{(p,k)}\right)}{\varphi\left(\frac{k}{(p,k)}\right)} \tag{14.8}$$

That's all. Not sure what more to say at this point.

Note. TODO – The Hurwitz zeta can be avoided entirely by working directly with the functional equation for the L-functions, as given by Apostol[**?**, Chapter 12, Theorem 12.11]. The direct form seems to imply some sort of result/constraint on the $p \neq 1$ terms in the expansion. It also suggests that most of the derivation above could be made clearer by assuming a generic functional equation, and stating results in terms of that. (e.g. assume Selberg-class type functional equation). The only tricky part is understanding the asymptotic behavior of zeta for large s.

15. Review of Dirichlet Series

The following sections consider sums with $\zeta(s)$ in the denominator. There are many Dirichlet series that achieve this. The canonical one, involving the Mobius function is

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \tag{15.1}$$

But there is also one for the Euler Phi function:

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \tag{15.2}$$

The Liouville function:

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} \tag{15.3}$$

The von Mangoldt function

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} \tag{15.4}$$

There are a dozen others that can be readily found in introductory textbooks and/or the web.

One may construct in a very straightforward way the Dirichlet series for $1/\zeta(s-a)$ for any complex a, as well as $1/\zeta(2s-a)$. These can be constructed by a fairly trivial application of Dirichlet convolution, together with the Mobius inversion formula. In short, one has an old, general theorem that

$$\sum_{n=1}^{\infty} \frac{(f*g)(n)}{n^s} = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \sum_{m=1}^{\infty} \frac{g(m)}{m^s} \tag{15.5}$$

where $f * g$ is the Dirichlet convolution of f and g:

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) \tag{15.6}$$

Since Dirichlet convolution is invertible whenever $f(1) \neq 1$ (and/or $g(1) \neq 1$), one may multiply and divide Dirichlet series with impunity, more or less.

I haven't yet seen a way of building $1/\zeta(\alpha s + \beta)$, or more complicated expressions.

Perhaps the most important generalization is that for the Dirichlet L-functions, since these are the ones for which the GRH applies. Specifically, one has

$$\sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} = \frac{1}{L(s,\chi)} \tag{15.7}$$

where χ is the Dirichlet character.

16. Numeric Exploration of Dirichlet Series

This section provides a numeric exploration of the finite differences of the various Dirichlet series given above. Consider first

$$d_n = \sum_{k=2}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{\zeta(k)} \tag{16.1}$$

After a numerical examination of d_n in the range of $2 \leq n \leq 1000$, one would be tempted to incorrectly conclude that $\lim_{n\to\infty} d_n = 2$; this numerical behavior is shown in the graphs 16.1 and 16.2. Exploring numerically into the higher range $1000 \leq n \leq 50000$, one discovers that d_n is oscillatory. The explanation for this behavior is presented below.

The asymptotic behavior of the d_n can be obtained by the corresponding Norlund-Rice integral. That is, one writes

$$d_n = \frac{(-1)^{n-1}}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} \frac{1}{\zeta(s)} \frac{n!}{s(s-1)(s-2)\cdots(s-n)} ds \tag{16.2}$$

The contour may be closed to the left, and the Cauchy-Riemann theorem applied. The integral has poles at zero, at the "trivial zeroes" $-2k$, and at the non-trivial zeroes $\rho = \sigma + i\tau$. Thus, one has

$$d_n = -\frac{1}{\zeta(0)} + \sum_{k=1}^{\infty} \frac{n!\,(2k-1)!}{(2k+n)!} \frac{1}{\zeta'(-2k)} + c_n \tag{16.3}$$

where the non-trivial zeros contribute

$$c_n = (-1)^n \sum_{\rho} \frac{n!}{\rho(\rho-1)\cdots(\rho-n)} \frac{1}{\zeta'(\rho)}$$

Substituting for

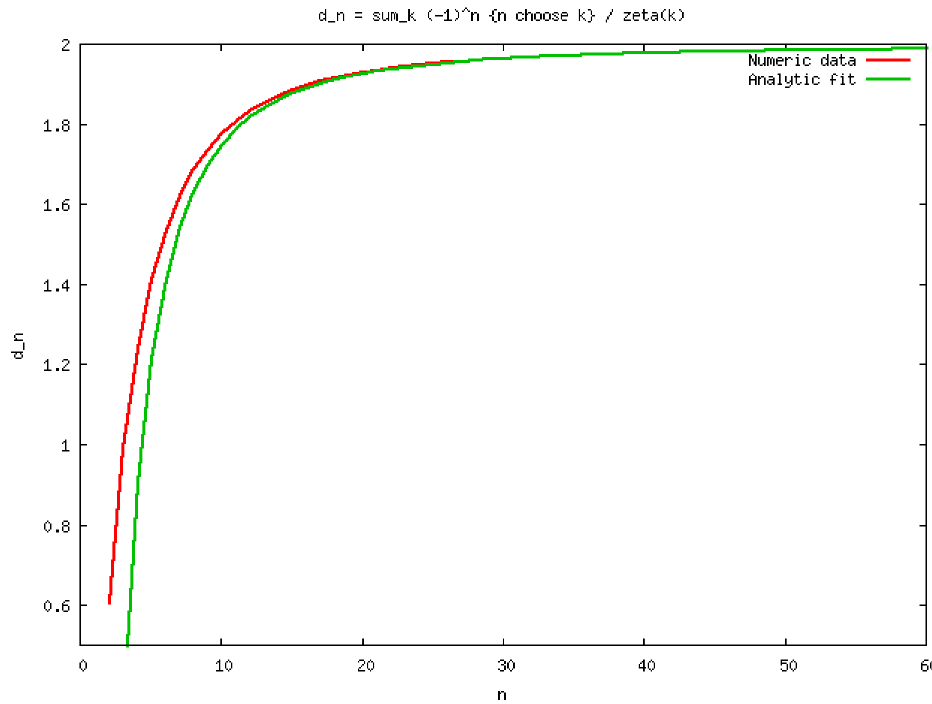
$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1)$$

one has

$$d_n = -\frac{1}{\zeta(0)} + \sum_{k=1}^{\infty} (-1)^k \frac{n!}{(2k+n)!} \frac{(2\pi)^{2k}}{k\zeta(2k+1)} + c_n \tag{16.4}$$

The value of zeta at zero, $\zeta(0) = -1/2$ accounts fully for the low-n behavior seen in the numerical sums. The next few terms in the k summation provide corrections that die

FIGURE 16.1. Graph of d_n for smaller n



The red graphic shows the numerically computed values for

$$d_n = \sum_{k=2}^{n} (-1)^k \binom{n}{k} \frac{1}{\zeta(k)}$$

in the range of $2 \leq n \leq 60$. It strongly behavior suggests an asymptotic approach to $d_n \to 2$. This numeric conclusion is incorrect, as demonstrated in the text. The small-n behavior is reasonably approximated by the analytic form

$$2 - \frac{4\pi^2}{(n+1)(n+2)\zeta(3)} = 2 - \frac{32.842386\cdots}{(n+1)(n+2)}$$

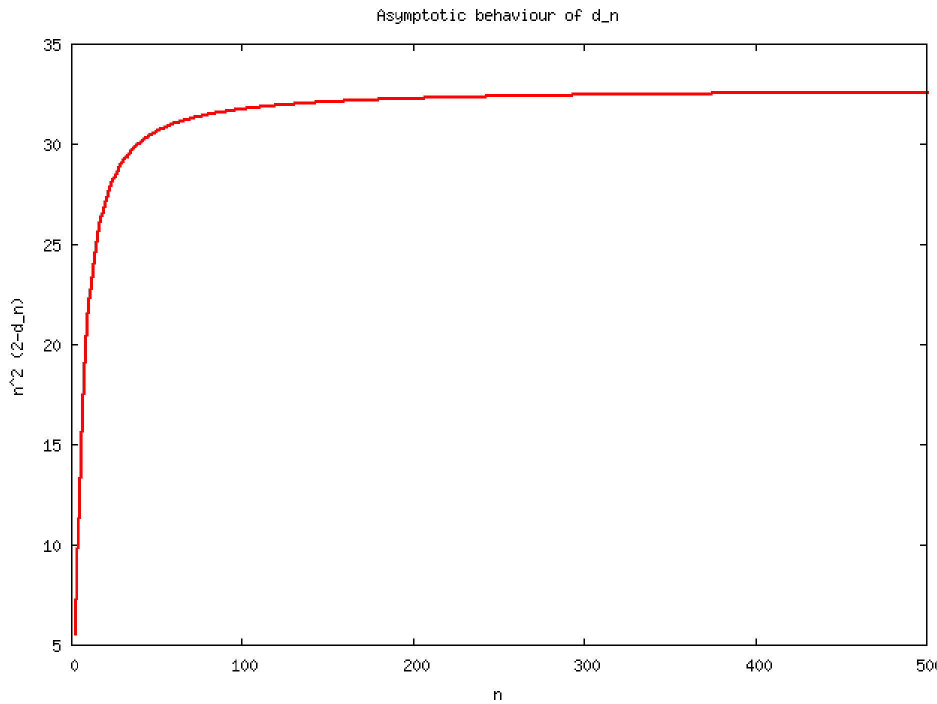
which is graphed as the green line.

away rapidly for large n, and thus aren't particularly interesting. The contribution of the non-trivial zeros is more surprising. One has

$$c_n = -\sum_{\rho} \frac{\Gamma(n+1)\Gamma(-\rho)}{\Gamma(n-\rho+1)} \frac{1}{\zeta'(\rho)} \tag{16.5}$$

For large n, the Stirling approximation may be applied. The sum has two distinct regimes. Write $\rho = \sigma + i\tau$ with $\sigma = 1/2$ if the Riemann Hypothesis is assumed. One may then consider the sum for the case where $\tau \ll n$, and the case where this does not hold. Writing $c_n = \alpha_n + \beta_n$ with α_n for the first few terms of the sum, and β_n for the remaining terms,

FIGURE 16.2. Graph of d_n for intermediate n



The above figure shows a graph of $n^2(2-d_n)$ in the range of $2 \le n \le 500$. As with the previous graphic, it strongly but incorrectly suggests that $d_n \to 2$ in the limit of large n. That this is not the case can be discovered by pursuing larger n. The value being approached is $4\pi^2/\zeta(3) = 32.842386\ldots$

one then has, after applying the Stirling approximation, that

$$\alpha_n = -\left[1+\mathcal{O}\left(\frac{1}{n}\right)\right] \sum_{\tau \ll n} (n+1)^\sigma \frac{\Gamma(-\sigma - i\tau)}{\zeta'(\sigma+i\tau)} e^{i\tau \log(n+1)} \tag{16.6}$$

Assuming the Riemann hypothesis, so that $\sigma = 1/2$, one has that $\alpha_n = \mathcal{O}(\sqrt{n})$. This is suppressed by a small factor, since $\left|\Gamma(\frac{1}{2}+i\tau)\right| \sim e^{-\tau\pi/2}$. Clearly, α_n is also slowly oscillatory. The remaining terms take the form

$$\begin{aligned}
\beta_n &= -\left[1+\mathcal{O}\left(\frac{1}{n}\right)\right] \sqrt{2\pi} \sum_{\tau \gtrsim n} \sqrt{\frac{\tau+in}{\tau n}} \times \\
&\quad \exp\left[\frac{n+1}{2}\log\left(\frac{n^2}{n^2+\tau^2}\right) + \frac{\sigma}{2}\log\left(1+\frac{n^2}{\tau^2}\right) - \tau \arctan\frac{n}{\tau}\right] \times \\
&\quad \exp i\left[(n+1)\arctan\frac{\tau}{n} + \frac{\tau}{2}\log\left(1+\frac{n^2}{\tau^2}\right)\right] \times \frac{1}{\zeta'(\rho)}
\end{aligned}$$

Since one has $\log n^2/(n^2+\tau^2) < 0$, one has that the remaining terms in the sum are exponentially small, provided that $\zeta'(\rho)$ never becomes arbitrarily small. Since the number of non-trivial zeros of the Riemann zeta does not increase exponentially, the sum can be estimated, and one concludes that

$$\beta_n = \mathcal{O}\left(e^{-Kn}\right) \tag{16.7}$$

for some $K > 0$. The value of K can be bounded away from zero, simply by arranging which terms are grouped into the finite sum α_n, and which are not. Thus, for large n, the contribution of β_n can be ignored, and the asymptotic form of d_n is given by

$$d_n = 2 + \alpha_n + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$$

where the Riemann hypothesis is assumed, so that a term $\sqrt{n}$ can be brought out of the α_n. Since one has $\left|\Gamma(\frac{1}{2} + i\tau)\right| \sim e^{-\tau\pi/2}$, each term contributing to α_n is considerably smaller than the last. Summing together the zeros above and below the real axis, one has

$$d_n = 2 - 2\sqrt{n+1} \sum_{n \gg \tau > 0} \Re\left[\frac{\Gamma\left(-\frac{1}{2} - i\tau\right) e^{i\tau\log(n+1)}}{\zeta'\left(\frac{1}{2} + i\tau\right)}\right] \tag{16.8}$$

The leading contribution to α_n is given by the first zero at $\rho = 0.5 \pm i14.1347251417\ldots$, followed by $\rho = 0.5 + i21.02203963877$. Then one has

$$\Gamma\left(-0.5 - i14.1347251417\right) = 4.036348365 \times 10^{-11} \times e^{-i\,2.850233468}$$

$$\Gamma\left(-0.5 - i21.02203963877\right) = 5.436324603 \times 10^{-16} \times e^{i\,2.572808318}$$

$$\zeta'\left(0.5 + i14.1347251417\right) = 0.783296 + i\,0.1247$$

$$\zeta'\left(0.5 + i21.02203963877\right) = 1.109295 - i\,0.2487297$$

Taking only the contribution from the first zero, one has

$$d_n \approx 2 - \sqrt{n+1}\left[1.01779 \times 10^{-10} \cos\left(14.1347 \log\left(n+1\right) - 3.0081\right)\right] \tag{16.9}$$

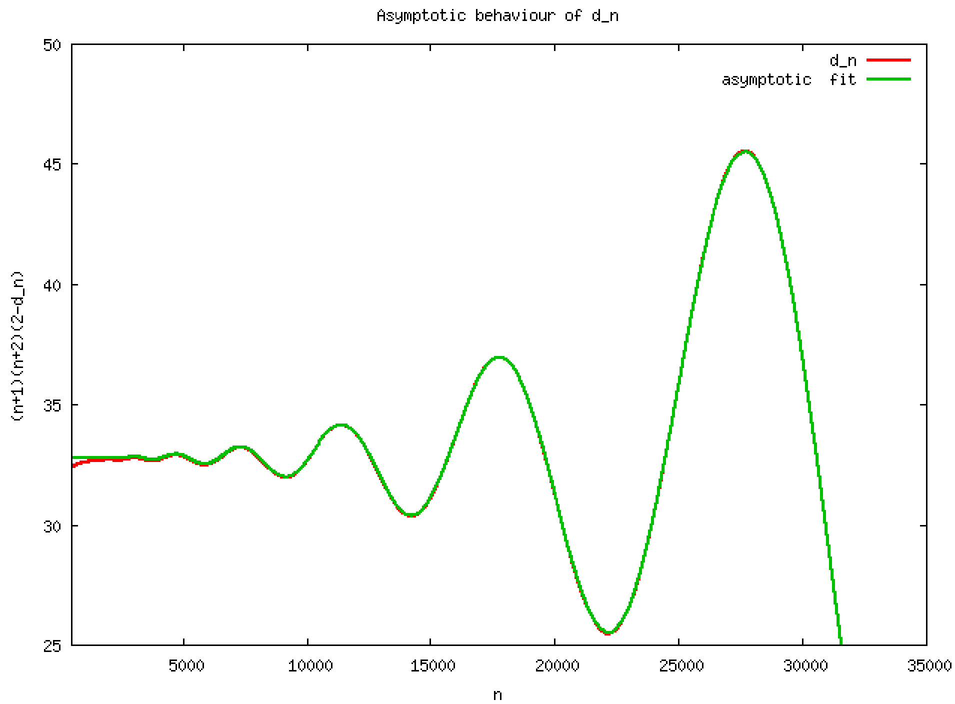
which may be seen to fit the data very well, as shown in figure 16.3.

16.1. **Equivalence to the Riemann Hypothesis.** What is curious about this result is that, if the Riemann Hypothesis holds, then there are no further corrections to this asymptotic behavior. The first zero gives the dominant term, and the second and subsequent zeroes give corrections that are at least 10^{-5} smaller, but of the same order in n. If there is a non-trivial zero that is not on the critical line, then it will contribute to equation 16.6 with a value of $\sigma_{broke} \neq 1/2$, and so one will instead have $d_n = \mathcal{O}\left(n^{\sigma_{broke}}\right)$. However, the contribution of such a zero would be strongly suppressed, by a value of $e^{-\pi\tau/2}$. Since the Riemann hypothesis has been verified to $\tau \sim 10^{12}$, the deviation from the square-root behavior would be numerically inaccessible. At any rate, the above sketch demonstrates that $d_n = \mathcal{O}\left(\sqrt{n}\right)$ is equivalent to the Riemann Hypothesis.

This conclusion generalizes almost trivially to the Generalized Riemann Hypothesis for the Dirichlet L-functions. Each of the equations in the previous section hold under the substitution of $L'(s,\chi)$ for $\zeta'(s)$. The only tricky step of the proof, which was glossed over and left unsupported above, was the assumption that $\zeta'(\rho)$ is never small, that is, that its bounded away from zero, so that the argument about the non-importance of the β_n holds. Provided that one can show this (and similarly, that $L'(s,\chi)$ is never small), then the argument follows through for the Generalized Riemann Hypothesis as well.

The above casual derivation can be inverted to show that the Riemann Hypothesis implies that the derivatives are bounded away from zero. It may be rigirously shown[**?**] that the RH is equivalent to having $d_n = \mathcal{O}\left(n^{1/2+\epsilon}\right)$ for all $\epsilon > 0$. To reconcile this with the

FIGURE 16.3. Asymptotic behavior of d_n



This graphic charts the value of $(n+1)(n+2)(2-d_n)$ in the range of $500 \leq n \leq 33000$. Rather than approaching a limit for large n, there are a series of oscillations that grow ever larger. The red line shows the numerically evaluated value of d_n, while the green line graphs the analytically derived

$$d_n \approx 2 - \sqrt{n+1}\left[1.01779 \times 10^{-10} \cos\left(14.1347 \log\left(n+1\right) - 3.0081\right)\right]$$

As can be seen, the fit is excellent.

above sketch that $d_n = \mathcal{O}\left(\sqrt{n}\right)$, one must conclude that, for all but possibly a finite set of zeroes ρ, one must have

$$\left|\zeta'(\rho)\right| > \mathcal{O}\left(e^{-\tau\pi/2}\right)$$

as otherwise the estimates for equation 16.7 would fail.

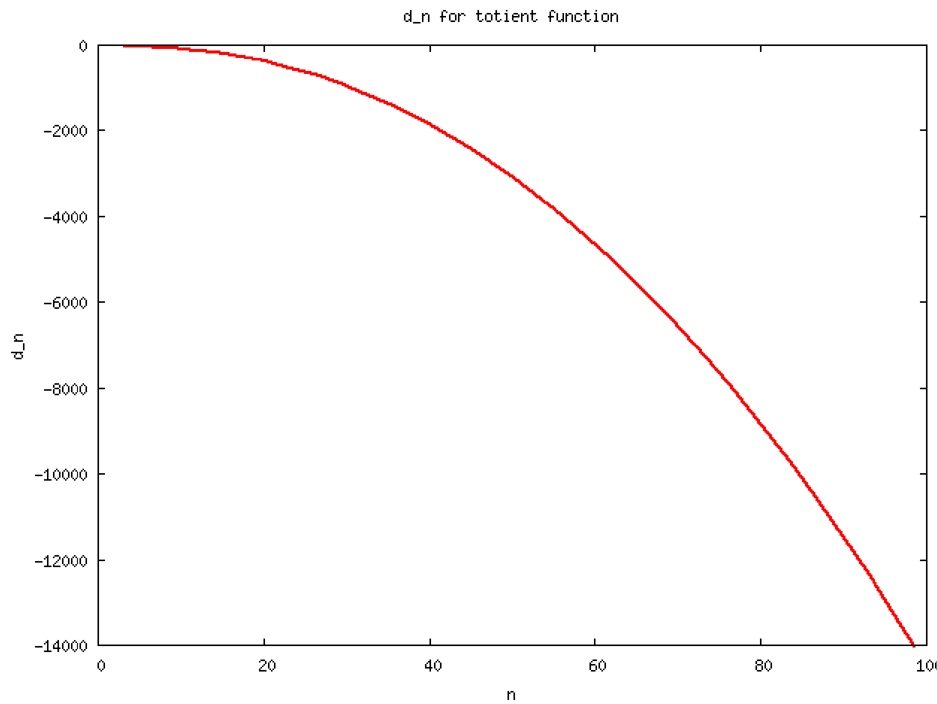
16.2. **Totient series.** The finite differences for the other series appear to show a similar pattern, except that the scale of the leading order is different. Consider, for example,

$$d_n^{\varphi} = \sum_{k=3}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array}\right) \frac{\zeta(k-1)}{\zeta(k)} \tag{16.10}$$

with the superscript φ indicating that the corresponding Dirichlet series involves the totient function φ. For smaller values of n, numeric analysis suggests that $d_n^{\varphi} = \mathcal{O}\left(n^2 \log n\right)$, as shown in graph 16.4.

The integrand of the corresponding Norlund-Rice integral is regular at $s=1$ and has a simple pole at $s=0$, and a double pole at $s=2$. In most other respects, it resembles the

FIGURE 16.4. Graph of d_n^φ



A graph of d_n^φ for $3 \leq n \leq 100$ shows rapidly increasing behavior.

integral for the Mobius series. The residue of the pol at $s = 0$ contributes

$$\text{Res}\,(s=0) = -\frac{\zeta(-1)}{\zeta(0)} = -\frac{1}{6}$$

while that at $s = 2$ contributes

$$\begin{aligned}\text{Res}\,(s=2) &= \Gamma\left(n+1\right)\frac{d}{ds}\left[\frac{(s-2)^2\zeta(s-1)\Gamma(-s)}{\zeta(s)\Gamma(n-s+1)}\right]\Bigg|_{s=2} \\ &= \frac{3n(n-1)}{\pi^2}\left[\frac{3}{2}-\gamma+\log\left(2\pi\right)-12\log A-\psi(n-1)\right]\end{aligned}$$

Here, the constant A is the Glaisher-Kinkelin constant $A = 1.28242712910062263687534256886979\ldots$. and $\psi(n)$ is the digamma function, which is just the harmonic number at the integers:

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k}$$

As before, the residues of the trivial zeros contribute $\mathcal{O}(1/n^2)$ to the sum. These are

$$\text{Res}\,(s<0) = \frac{n!}{2\pi^2}\sum_{k=1}^{\infty}\frac{(2k+1)!}{(2k+n)!}\,\frac{\zeta(2k+2)}{k\zeta(2k+1)}$$

Thus, the analytic development indicates that

$$d_n^{\varphi} = \text{Res}\,(s \le 2) - \sum_{\rho} \frac{\Gamma(n+1)\Gamma(-\rho)}{\Gamma(n-\rho+1)}\,\frac{\zeta(\rho-1)}{\zeta'(\rho)} \tag{16.11}$$

for the series.

It seems that, at least for the small zeroes, one has that $\zeta(\rho-1) \sim 1$, and so, just as in the Mobius function sums, one has oscillatory behavior given by

(16.12)

$$d_n^{\varphi} - \text{Res}\,(s \le 2) = 2\sqrt{n+1} \sum_{n \gg \tau > 0} \Re\left[\Gamma\left(-\frac{1}{2} - i\tau\right) e^{i\tau \log(n+1)} \frac{\zeta\left(-\frac{1}{2}+i\tau\right)}{\zeta'\left(\frac{1}{2}+i\tau\right)}\right]$$

where, for the first zero,

$$\zeta\,(-0.5 + i14.1347251417) = -1.184474313 - i\,0.3142933325$$

so that the oscillations are given by

(16.13)

$$d_n^{\varphi} - \text{Res}\,(s \le 2) \approx \sqrt{n+1}\left[1.247262 \times 10^{-10} \cos\left(14.1347 \log\left(n+1\right) - 5.89033\right)\right]$$

A graph comparing the two sides of this equation is shown in figure 16.5.

16.3. **Liouville series.** Let

$$d_n^{\lambda} = \sum_{k=2}^{n} (-1)^k \binom{n}{k} \frac{\zeta(2k)}{\zeta(k)} \tag{16.14}$$

be the series corresponding to the Liouville function. The integrand of the corresponding Norlund-Rice integral is regular at $s=1$ and has a simple pole at $2s=1$ and another at $s=0$. In other respects, it resembles the integral formulation for the Mobius series. The simple pole contributes a term of $\mathcal{O}(\sqrt{n})$. Curiously, there are no poles from the trivial zeros, as these are balanced out by the numerator. The contribution of the non-trivial zeroes to the sum may be estimated as in the Mobius sum.

More precisely, the residue of the pole at $s=0$ provides a term

$$\text{Res}\,(s=0) = -1$$

The residue of the pole at $s=1/2$ provides a term

$$\text{Res}\left(s = \frac{1}{2}\right) = -\frac{\sqrt{\pi}}{\zeta\left(\frac{1}{2}\right)}\,\frac{\Gamma\left(n+1\right)}{\Gamma\left(n+\frac{1}{2}\right)}$$

where

$$\zeta\left(\frac{1}{2}\right) \approx -1.4603545088095868128849991 5252\ldots$$

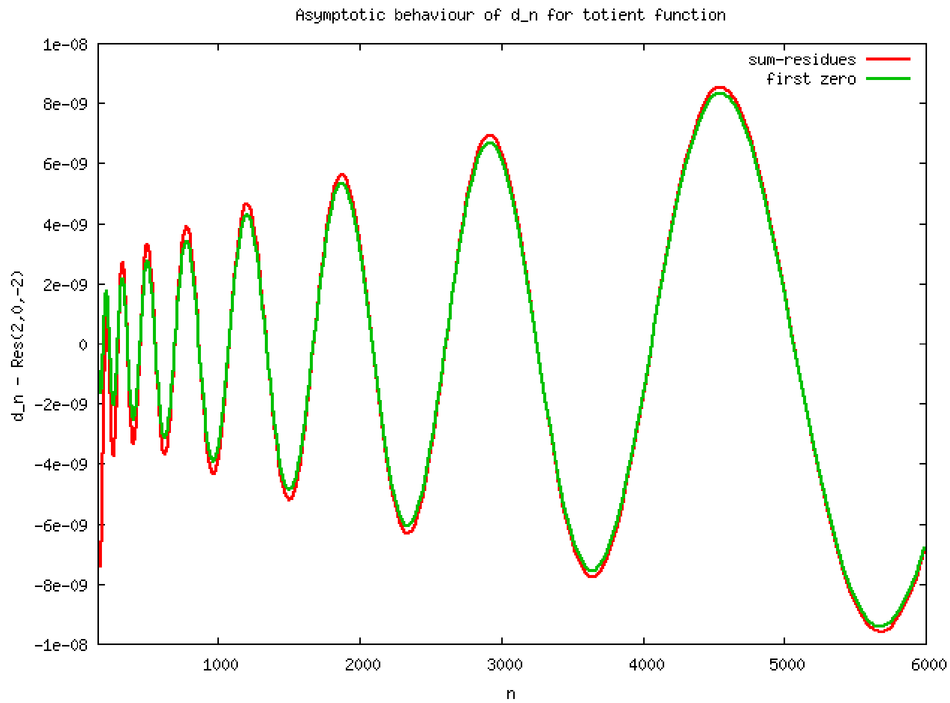
This residue accounts very well for the behavior of d_n^{λ}, as demonstrated in figure 16.6.

The full behavior of the d_n^{λ} should be given by

$$d_n^{\lambda} = -\frac{\sqrt{\pi}}{\zeta\left(\frac{1}{2}\right)}\,\frac{\Gamma\left(n+1\right)}{\Gamma\left(n+\frac{1}{2}\right)} - \sum_{\rho} \frac{\Gamma(n+1)\Gamma(-\rho)}{\Gamma(n-\rho+1)}\,\frac{\zeta(2\rho)}{\zeta'(\rho)} \tag{16.15}$$

Applying the Stirling approximation, one obtains

FIGURE 16.5. Asymptotic behavior of d_n^{φ}



This graphic shows the asymptotic behavior of d_n^{φ} after the leading residues are subtracted. The red line shows

$$d_n^{\varphi} - \text{Res}\,(s=2) - \text{Res}\,(s=0) - \text{Res}\,(s=-2)$$

The green line shows the contribution of the first non-trivial zero, that is, the green line shows

$$\sqrt{n+1}\left[1.247262 \times 10^{-10} \cos\left(14.1347 \log\left(n+1\right) - 5.89033\right)\right]$$

As may be seen, the match is excellent.

$$\begin{aligned} d_n^{\lambda} &= \text{Res}\,(s=0) + \text{Res}\left(s=\frac{1}{2}\right) \\ & -2\sqrt{n+1} \sum_{n \gg \tau > 0} \Re\left[\Gamma\left(-\frac{1}{2} - i\tau\right) e^{i\tau \log(n+1)} \frac{\zeta\left(1+2i\tau\right)}{\zeta'\left(\frac{1}{2}+i\tau\right)}\right] \end{aligned}$$

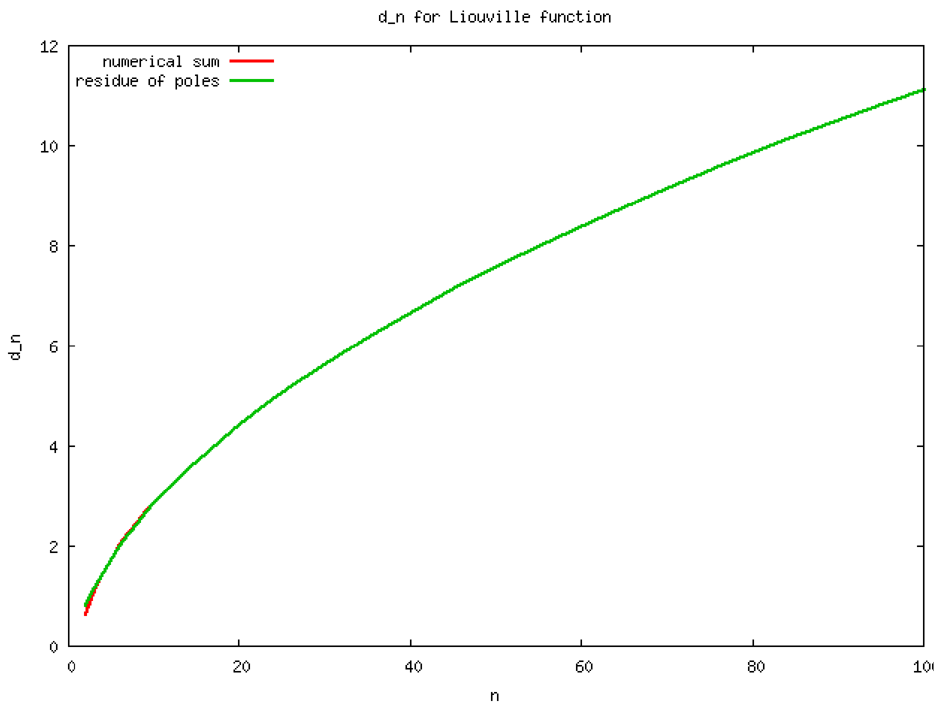
Using

$$\zeta\left(1 + i2 \times 14.1347251417\right) = 1.836735353402834 - i0.6511975965222686$$

one then expects

(16.16)
$$d_n^{\lambda} \approx \text{Res}\,(s=0) + \text{Res}\left(s=\frac{1}{2}\right) + \sqrt{n+1}\left[1.98342 \times 10^{-10} \cos\left(14.1347 \log\left(n+1\right) - 3.3488\right)\right]$$

FIGURE 16.6. d_n^λ for the Liouville function



This figure shows the basic behavior for the finite differences d_n^λ corresponding to the Liouville function, plotted in red. For comparison, plotted in green, is the value of

$$-1-\frac{\sqrt{\pi}}{\zeta\left(\frac{1}{2}\right)}\frac{\Gamma\left(n+1\right)}{\Gamma\left(n+\frac{1}{2}\right)}$$

Clearly, this provides an excellent fit for the low-order behavior of the series.

A graph of this is shown in figure 16.7; again, the fit is excellent.

As is the case for the Mobius sums, the asymptotic behavior of $d_n^\lambda=\mathcal{O}(\sqrt{n})$ implies and is implied by the Riemann hypothesis, provided that the coefficient $\zeta\left(1+2i\tau\right)/\zeta'\left(\frac{1}{2}+i\tau\right)$ is bounded away from zero.

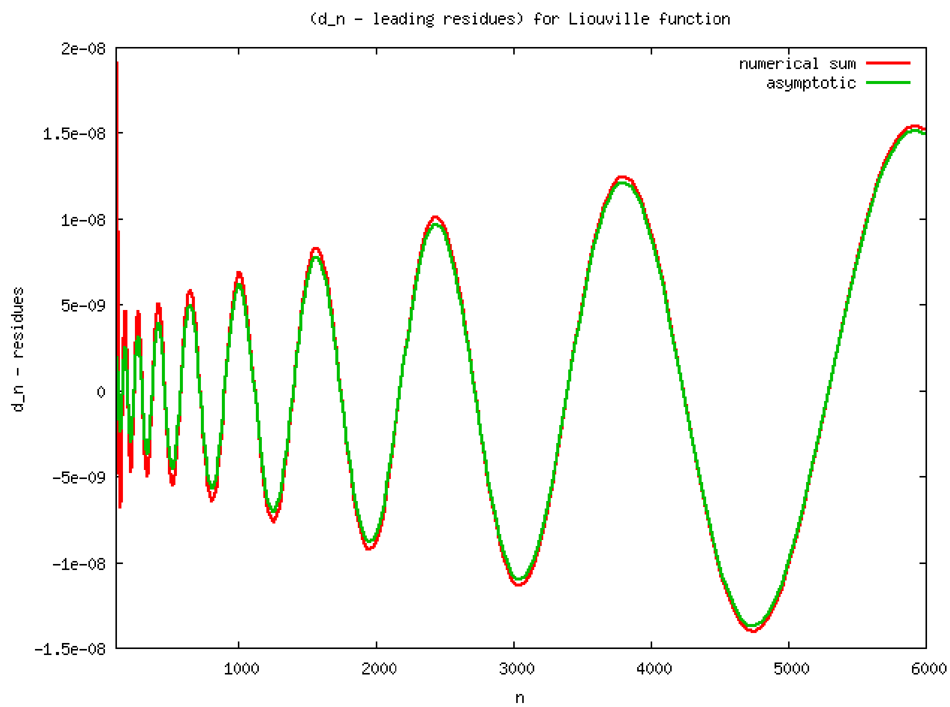
17. RELATION TO LITERATURE

The following is a list of Newton series or other suggestive sums or integrals that occur in the literature, both modern and older. Many of these may be amenable to the techniques given above, or have results that follow immediately from the above.

17.1. **Reciprocal Riemann Zeta.** The Riemann zeta function has regularly-spaced zeros along the negative real axis. Thus, the reciprocal has poles at the (even) integers, and thus resembles the Norlund-Rice integrand. Viz:

$$\oint_C\frac{ds}{\zeta(s)}\sim\sum\left(\begin{array}{c}n\\k\end{array}\right)\frac{1}{\zeta'(2n)}$$

FIGURE 16.7. Asymptotic form of d_n^λ



This figure shows a graph of the series d_n^λ with the residue of the pole at $s = 1/2$ removed. That is, the red line shows

$$d_n^\lambda + 1 + \frac{\sqrt{\pi}}{\zeta\left(\frac{1}{2}\right)} \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}$$

for the range of $110 \leq n \leq 6000$. The green line is a graph of

$$\sqrt{n+1}\left[1.98342 \times 10^{-10} \cos\left(14.1347 \log(n+1) - 3.3488\right)\right]$$

As can be seen, the fit is very good, and improves for large n.

where the contour encircles n poles. Not clear how to turn the integral into something suitable for a saddle-point method.

17.2. **Maslanka/Baez-Duarte.** A similar sum appears in [**?**] as

$$c_n = \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array}\right) \frac{1}{\zeta(2k+2)}$$

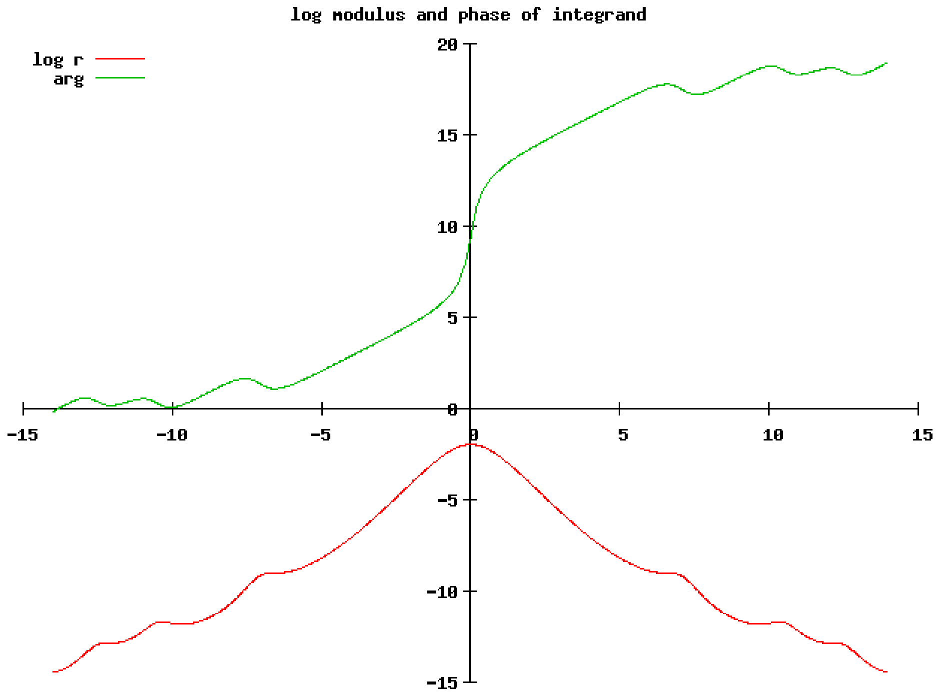
and furthermore, it is claimed that

$$c_n \ll n^{-3/4+\epsilon} \forall \epsilon > 0$$

is equivalent to RH. Note the summand is just a ratio of Bernoulli numbers and powers of π. The NR integral is

$$c_n = \frac{(-1)^n}{2\pi i} \int_{-1/4-i\infty}^{-1/4+i\infty} \frac{1}{\zeta(2s+2)} \frac{n!}{s(s-1)\cdots(s-n)} ds + \frac{\delta_{n0}}{2}$$

FIGURE 17.1. Log integrand



Graph of log and arg of integrand for $n = 6$. The bumps at 7,11,13 correspond to Riemann zeros at 14, 21, 26. For large n, the real part does not become more parabolic, but retains roughly the same shape. However, for large n, the phase runs more rapidly, pushing apart the hilltops.

The $n = 0$ Cauchy integral has an contribution of $1/2$ coming from the semi-circular contour at infinity on the right, which vanishes for $n \neq 0$. The integrand is suggests a saddle point, bounded by poles at $s = 0, -2$ while getting obviously small for large imaginary s. The problem is the the pole at $s = 0$ has a residue of opposite sign from that at $s = -2$ and, so, if we are lucky, there is an inflection point between these two locations, viz. a point where the first derivative vanishes.

I'm confused at this point. It appears that I can push the contour at $\sigma = -1/4$ further to the left, which amazingly passes me through the critical strip without changing the value of the integral! I guess that the contribution of all of residues of the poles at the zeros of ζ must add up to zero. I didn't know this, I presume it follows trivially(??) by complex conjugation.

17.3. **Hasse-Knoppe.** Helmut Hasse and Conrad Knoppe (1930) give a series for the Riemann zeta is convergent everywhere on the complex s-plane, (except at $s = 1$):

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s}$$

It would be curious to explore the associated integral. I'm particularly intrigued by the power-of-2 sum.

17.4. **Hasse for Hurwitz.** Hasse also gave a similar, globally convergent, expansion for the Hurwitz zeta:

$$\zeta(s,q)=\frac{1}{s-1}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\left(\begin{array}{c}n\\k\end{array}\right)(q+k)^{1-s}$$

Same question as above.

17.5. **Dirichlet Beta.** The Dirichlet beta is given by

$$\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

and has a functional equation

$$\beta(s)=\left(\frac{\pi}{2}\right)^{s-1}\Gamma(1-s)\cos\frac{\pi s}{2}\,\beta(1-s)$$

This is just the L-function for the second character modulo 4, so we already have this sum, and can read it right off. XXX Do this.

17.6. **L-function of principal character modulo 2.** The L-function of the principal character modulo 2 is given by

$$L(\chi_1,s)=\left(1-2^{-s}\right)\zeta(s)$$

and we already have expansions for this. The Newton series for this appears in [**?**] as equation 4.11 and also in [**?**] equation 16 as the sum

$$S_1(n)=\sum_{k=2}^{n}(-1)^k\left(\begin{array}{c}n\\k\end{array}\right)\left(1-2^{-k}\right)\zeta(k)$$

Coffey states the theorem that

$$S_1(n)\geq\frac{n}{2}\ln n+(\gamma-1)\,\frac{n}{2}+\frac{1}{2}$$

We can read off the full result instantly from the result on the L-functions. XXX Do this.

17.7. **Another Coffey sum.** Coffey [**?**] shows interest in another sum:

$$S_3(n)=\sum_{k=2}^{n}(-1)^k\left(\begin{array}{c}n\\k\end{array}\right)2^k\zeta(k)$$

The reason for the interest in this sum is unclear.

17.8. **A Li Criterion-related sum.** Bombieri [**?**], Lagarias [**?**] and Coffey [**?**] provides a sequence $\eta_k$which seems to be of the form $\exp -k$ and thus suggests that the saddle-point techniques should be applicable. These appear in Lagarias equation 4.13 and in Coffey equation 10 as

$$\lambda_n=-\sum_{k=1}^{n}\left(\begin{array}{c}n\\k\end{array}\right)\eta_{k-1}+S_1(n)+1-\frac{n}{2}\left(\gamma+\ln\pi+2\ln 2\right)$$

where $S_1(n)$is given above, and λ_n are the Li coefficients

$$\lambda_n=\frac{1}{(n-1)!}\;\frac{d^n}{ds^n}s^{n-1}\ln\xi(s)\bigg|_{s=1}$$

and

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

Part of what is curious is that the η_k appear in an expression similar to the what is seen for the Stieltjes constants, but involve the von Mangoldt function.

17.9. **Prodinger, Knuth.** Prodinger considers a curious sum, and provides an answer; Knuth [**?**] had previously provided a related sum. Doesn't seem to be much to do here, as the leading terms are already given, and, from the comp-sci point of view, these are enough. The motivation for providing the exponentially small terms is uncertain. The Prodinger sum is [**?**]:

$$S=\sum_{k=1}^{n-1}\left(\begin{array}{c}n\\k\end{array}\right)\frac{B_k}{2^k-1}\simeq-\log_2 n+\frac{1}{2}+\delta_2\left(\log_2 n\right)$$

where B_k are the Bernoulli numbers. The Norlund-Rice integral is

$$S=\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{(-1)^n n!}{((s-1)(s-2)\cdots(s-n)}\cdot\frac{\zeta(1-s)}{2^s-1}\,ds$$

The poles due to 2^z-1 lead to the curious term in the asymptotic expansion:

$$\delta_2(x)=\frac{1}{\log 2}\sum_{k\neq 0}\zeta\left(1-\chi_k\right)\Gamma\left(1-\chi_k\right)e^{2\pi ikx}$$

where

$$\chi_k=\frac{2\pi ki}{\log 2}$$

The Knuth sums are similar, but different.

17.10. **Lagarias.** Lagarias [**?**] has a a sum over the Hurwitz zeta, equation 5.5:

$$T(n,z)=\sum_{k=1}^{n}(-1)^k\left(\begin{array}{c}n\\k\end{array}\right)\frac{1}{2^k}\zeta(k,z+1)$$

We've already computed this sum, so should be able to read off an answer directly, and improve significantly on Lagarias results. XXX Do this. XXX Actually, not quite ... the best from above is

$$A_n(m,2)=\sum_{k=2}^{n}(-1)^k\left(\begin{array}{c}n\\k\end{array}\right)\frac{1}{2^k}\zeta\left(k,\frac{m}{2}\right)$$

which misses the general z. I believe saddle point analysis can be generalized to handle the Lagarias sum.

18. Conclusions/Comments

Note. Not sure about the following ... interesting and suggestive but don't see where it leads.

The paper concludes by noting the general resemblance of the asymptotic behavior to that of two other famous number-theoretic sequences. The partition function $p_n$is given by Euler's infinite product[**?**]:

$$\sum_{n=0}^{\infty}p_n x^n=\prod_{m=1}^{\infty}\frac{1}{1-x^m}$$

and for large n, takes on the asymptotic form

$$p_n\sim\frac{\exp K\sqrt{n}}{4n\sqrt{3}}$$

with $K = \pi\sqrt{2/3}$. This form was given by Hardy and Ramanujan [ref] and Uspensky [ref], with a full series valid to all orders given by Rademacher[ref]. There is a similar asymptotic form for the Fourier coefficients of the modular j-invariant,

$$j(\tau) = e^{-2\pi i\tau} + \sum_{n=0}^{\infty} c_n e^{2\pi i n\tau}$$

where the coefficients take on the asymptotic values

$$c_n \sim \frac{\exp 4\pi\sqrt{n}}{n^{3/4}\sqrt{2}}$$

which was given by Petersson [ref][**?**]. The appearance of an asymptotic factor in the form of $e^{K\sqrt{n}}$ here and in the Newton series is suggestive. However, unlike the p_n and the c_n, the values $1/b_n$ are not integers; nor is the Fourier series $\sum_n x^n/b_n$ a modular form.

19. Appendix

A related but simpler integral can be found in Tom M. Apostol, *Introduction to Analytic Number Theory*, Lemma 3, Chapter 13[**?**]. The integral resembles equation 13.1. Re-expressed so as to heighten the resemblance, it states (XXX changed notation here too, this is probably/certainly wrong too.)

$$\begin{aligned} b_n(m) &= \frac{(-1)^{n+1}}{2\pi i}\, n! \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{1}{m^s s(s-1)\cdots(s-n)}\, ds \\ &= \frac{1}{2\pi i}\, n! \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{m^s}{s(s+1)\cdots(s+n)}\, ds \\ &= \begin{cases} \left(1-\frac{1}{m}\right)^n & \text{for } m \geq 1 \\ 0 & \text{for } m < 1 \end{cases} \end{aligned} \tag{19.1}$$

A naive application of this identity to the integrals in this paper leads to divergent formal sums. XXX clarify this statement.

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